MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 13, N° 1, 1984, pp. 57 — 63

ALGEBRAICAL STRUCTURES IN THE NETWORK THEORY

by

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A network is defined as being a finite, directed, loopless graph in which each arc has an associated measure. The measures associated to the arcs belong to a certain algebraical structure. The existence of an one-to-one map between a network and a square matrix having as elements the associated measures of the network, allows the transposition of an important class of problems from graph theory language in an algebrical language. The number of problems belonging to this class is large when the algebraical structure is very general.

The first matricial representation of a graph had a boolean character. The elements a_{ij} of the matrix A, associated to the graph, are defined as follows:

$$a_{ij} = \begin{cases} 1 \text{ if the arc } (i, j) \text{ exists} \\ 0 \text{ if it does not.} \end{cases}$$

The subject under discussion refers to general algebraical structures, defined axiomatically, which constitute measures for the arcs of the network in the broadest meaning possible. The measure of a path of the network is defined as being given by the product of the measures (in the algebraical structure meaning) attached to the arcs which make the respective path.

The first who defined an algebraical structure axiomatically and who used it for the study of paths in networks was GR. C. MOISIL [13] in 1960. His paper is the basis of an actual "waterfall" of generalizations and studies and the recent articles [1], [9], [23], [27] have proved that the subject under discussion has not been dealt with exhaustively.

Gr. C. Moisil defined an algebraical structure called a *semilatically ordered semigroup* as being a set S with the following properties:

In S it is defined an internal composition law „o" with the neutral element e, so that for any a, b, c of S hold:

$$a \circ b = b \circ a,$$
$$a \circ (b \circ c) = (a \circ b) \circ c,$$
$$a \circ e = e \circ a = a.$$

In S it is defined a partial order relation so that each pair of elements belonging to S should have a lower boundary in S. If one denotes $\subset$ the semiserial relation and $a \wedge b$ the lower boundary of the elements a and b then:

$a \subset a$,
$a \subset b$ and $b \subset a$ implies $a = b$,
$a \subset b$ and $b \subset c$ implies $a \subset c$,
$a \wedge b \subset a$, $a \wedge b \subset b$,
$c \subset a$ and $c \subset b$ implies $c \subset a \wedge b$.

In S, the laws of distributivity and absorbtion are valid:

$$a \circ (b \wedge c) = (a \circ b) \wedge (a \circ c),$$
$$a \wedge (a \circ b) = a.$$

Consider on S the square matricies $A = (a_{ij})$ and the matricial product $P = A \times B$, where $p_{ij} = \bigwedge_h (a_{ih} \circ b_{hj})$. It is proved that if A is an $n \times n$ matrix on S having the property $a_{ii} = e$, $i = 1, 2, \ldots, n$, then $A^{n-1} = A^n$. The author shows that certain problems of transport economy can be solved on the basis of the above results. Such problems are: the problem of determining the transport route on which the costs are minimum, the problem of determining the transport route with the greatest transport capacity and the problem of determining the route on which the probability of the transport outcome is maximum.

In 1961 M. YOELI [28] defined the algebrical structure which he called a *Q — semiring*. A Q — semiring is a set of elements Q which has two binary operations:[1] an additive operation „⊕" and a multiplicative operation „o" with the following properties:

$$a \oplus b = b \oplus a,$$
$$(a \oplus b) \oplus c = a \oplus (b \oplus c),$$
$$(a \circ b) \circ c = a \circ (b \circ c),$$
$$a \circ (b \oplus c) = (a \circ b) \oplus (a \circ c),$$
$$(b \oplus c) \circ a = (b \circ a) \oplus (c \circ a)$$

[1] Different authors use different notations for the two operations. In order to keep the accuracy in this paper we use „⊕" and „o" and the corresponding neutral elements are denoted by 0 and e.

and which contains the elements θ and e with the following properties:

$$a \oplus \theta = a,$$
$$a \circ \theta = \theta \circ a = \theta,$$
$$a \circ e = e \circ a = a,$$
$$a \oplus e = e.$$

As it seen, M. Yoeli does not postulate the comutativity of multiplication.

The author defines matrices and operations with matrices on his algebraical structure. It is denoted by A the link matrix of a graph and by T the transmission matrix. The author proves that $A^m = T$ for $m \geq n - 1$, where n is the order of the matrix A.

There are two kinds of matrices associated to a network. They are different only because in one case $a_{ii} = e$, $i = 1, 2, \ldots, n$, like with Moisil and in the second case $a_{ii} = \theta$, $i = 1, 2, \ldots, n$. In both cases if the arc (i, j) does not exist in the network, then $a_{ij} = \theta$.

The calculation of the powers of the matrix A associated to a network is very important because the elements of the matrix A^k provide informations refering to the measures of certain paths formed of *a*) at most k arcs if the matricial representation was made with $a_{ii} = e$ or *b*) exactly k arcs if the matricial representation was made with $a_{ii} = \theta$. The case $k = n - 1$ is also interesting.

In 1965 R. C. CRUON and P. HERVÉ [4] defined a particular algebraical structure referring to the longest path in a network which cannot be included in the previous structures as it does not satisfy the absorbtion property. In [15], V. PETEANU defined in 1967 an algebraical structure which includes all the path problems studied by Moisil, Yoeli, Cruon and Hervé without being more general than the semilaticially ordered semigroup or than the Q — semiring because the author presumed that the algebraical structure is a totally ordered set. The study of certain algorithms for the calculation of the powers of matrix A with $a_{ii} = e$, within an algebraical structure equivalent to that of Moisil, was made by I. TOMESCU in 1966 [25] and in 1968 [26]. As far as the Q — semiring is concerned, P. POBERT and J. FERLAND in 1968 [20] ellaborated calculation methods for A^k starting from a matrix for which $a_{ii} = \theta$.

In 1969 in [16] V. PETEANU defined a more general algebraical structure called *C — semigroup* or *routing semigroup*. This is a Q — semiring in which the condition $a \oplus e = e$ was replaced by $a \oplus a = a$.

The property of idempotence for addition is weaker than absorbtion because $a \oplus e = e$ implies $a \oplus a = a$ but not viceversa. (Such an algebraical structure was also called by M. L. DUBREIL-JACOTIN, L. LESIEUR and R. CROISOT [5] „*gerbier*" without having any connection with the study of paths in networks).

The C — semigroup is more general than the semilaticially ordered semigroup and then the Q — semiring, containing also the algebraical structure proposed by R. Cruon and P. Hervé.

In the same paper [16] the author studied a series of properties of the elements of a C — semigroup in detail. It was also defined the notion of p — *stability* as being a generalization of idempotence. An element a belonging to the C — semigroup is called p — *stable* if $a^p = a^{p+1}$. It is caled weak p — stable if:

$$a \oplus a^2 \oplus \ldots \oplus a^p = a \oplus a^2 \oplus \ldots \oplus a^{p+1}.$$

In the paper [17] the author defined matricies, the matricial operations, systems of equations, linear combinations etc ... on a C — semigroup. It was proved that the set of matricies of order n on a C — semigroup constitutes in its turn a C — semigroup. There were also studied conditions for the compatibility of systems of equations and also algorithmas for their solving in the case when the matrix associated to the network is of the type $a_{ii} = e$. The *stabilization operators* are defined and their properties are studied.

In 1971 B. A. CARRE [3] redescovered the C — semigroup for the particular case in which the multiplication ,,o" is commutative and he called this structure *semiring*. The optimal path problem in networks is presented in terms of linear algebra. The matrix associated to the network is of the type $a_{ii} = \theta$. Carré is the first who solves problems of path in network adapting to the semiring the Jacobi and Gauss-Seidel methods, the Gauss and Jordan elimination methods and an escalator method.

In 1975 M. GONDRAN [7], [8] generalized the C — semigroup giving up additive idempotency. Later in [9] he called this algebraical structure a *dioid* (according to J. KUNTZMANN [10]). Obviously, the range of matricially approached problems is widerred thus including path counting problems, shortest k path problems, ε — optimal path problems, Markov chains problems.

In [8] Gondran defined the p — *regular* element which is very much alike with the weak p — stable element given in [16]. Gondran used the associated matrix of the type $a_{ii} = \theta$. In order to calculate the powers of this matrix he generalized the algorithms given by Carré [3].

In the same year B. ROY [23] defined the *paths algebra*. This is an algebraical structure consisting of a set L and two operations $\mathfrak{C}$ and E called *concatentation* and *extraction* respectively. Concatenation is associative and has a unit element e and extraction is associative, commutative and has a zero element which is absorbant for $\mathfrak{C}$. The elements of L are *multisets* (*bag* or *tas*) and the operations are defined in a proper way. Concatenation presents a certain kind of distributivity relating to extraction. There is a certain equivalence between Roy's structure and Gondran's dioid. The specific language used

by Roy in his paths algebra aims to making easier the formulation of some problems as the shortest k path problems, the determining of efficient path problem, the enumeration of paths with a certain property P problems etc ....

In the paper [1] from 1975 R. C. BACKHOUSE and B. A. CARÉ noticing the relationship between the semiring and the regular expression algebra (see A. SALOMAA [24] and A. GINSBURG [6]) defined the *regular algebra* as being a C — semigroup in which a *closure operator* ,,*" is also defined having the following properties:

$$a^* = e \oplus a \circ a^*,$$
$$a^* = (e \oplus a)^*.$$

The results obtained by Carré [3] are extended to the regular algebra, special attention being given to calculation methods. It is shown that if $e \oplus a = e$, then $a^* = e$. Consequently, the closure operator is trivial and can be eliminated. The resulting algebra is Yoeli's Q — semiring. An interesting application of the regular algebra is given by A. MARTELLI [11] who determined the minimal sections between any two nodes of a network.

In 1976 M. MINOUX [12] sought a more general algebraical structure than the dioid so that it could also comprise the shortest path with time constrait problems. Thus he considered a set S endowed with an operation $\oplus$ which is associative, commutative and has a neutral element θ. He denoted with H the set of all endomorfisms of S relating to $\oplus$. The operation $\oplus$ induces on H an operation, also denoted by $\oplus$ and defined as follows:

$$(h \oplus g)(a) = h(a) \oplus g(a), \quad h, g \in H, \ a \in S.$$

A second operation denoted $\otimes$ is considered in H which is defined by $h \otimes g = g \circ h$ where ,,o" is the composition of applications operation. The $\otimes$ operation is associative, distributive relating to $\oplus$ and has an unit element. The structure $(S, H, \oplus, \otimes)$ is called by the author a *generalized routing algebraical structure*. The arcs of a network are measured with elements of H and a generalized matrix associated to a network is defined. A series of properties and results obtained for dioids is extended to the above mentioned structure. Some of the new problems which can be solved in generalized routing algebraical structure are: the shortest path with time depending lengths of arcs, the shortest k paths with time depending lengths of arcs, maximal probability paths with probabilities depending on time, etc ....

In 1979 WONGSEELASHOTE [27] introduced the concept of *path-spaces* using multisets as B. Roy [23]. The author denoted by N_∞^X the set of multisets with elements from a given monoid $(X, \circ)$. In N_∞^X he defined two operations $\uplus$ and $\circ$ called multisum and multiproduct respectively, both having neutral elements and which verify the properties of Carré's semiring. A set V is called a *hereditary semiring* of $N_\infty^\times$ if V is its semiring having the following properties:

$\{x\} \in V$ for any $x \in X$,

$A \in V$ and $B \subseteq A$ implies $B \in V$.

A *path-space* is an ordered quadruple $(X, \circ, V, r)$ where $(X, \circ)$ is a monoid and r is a *self-map* of V satisfying:

$$r(\Phi) = \Phi,$$
$$r(A \uplus B) = r(r(A) \uplus B),$$
$$r(A \circ B) = r(r(A) \circ B) = r(A \circ r(B)).$$

It is shown that each semiring generates at least a path-space. It is also shown how to obtain a semiring from a path-space.

The author's reason for using multisets and path-spaces was to be able to include all the problems of paths in network in an unified theory.

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