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L_p — APPROXIMATION BY LINEAR COMBINATION OF INTEGRAL BERNSTEIN-TYPE OPERATORS

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1. Introduction

May [5] and Rathore [6] have described the following linear combination of a sequence, $\{K_n\}$, of positive linear operators:

$$(1.1)\qquad L_n(f(t),\ k,\ x) = \sum_{j=0}^{k} c(j,\ k) K_{d_j n}(f(t),\ x),$$

where $d_0,\ d_1,\ \ldots,\ d_k$ are $k+1$ arbitrary, fixed and distinct positive integers and

$$(1.2)\qquad c(j,\ k) = \prod_{\substack{i=0\\ i\neq j}}^{k} \frac{d_j}{d_j - d_i},\quad k \neq 0,\ \text{and}\ c(0,\ 0) = 1.$$

Recently Sinha [7] proved both direct and inverse results for approximation in the space, $L_p(I)$ $(p \geq 1)$, of p-th power Lebesgue integrable functions on the interval $I = [0,\ 1]$ by means of the linear combination (1.1) of the wellknown Bernstein-Kantorovitch polynomials.

Durrmeyer [2] introduced the following integral version of the Bernstein polynomials:

$$(1.3)\qquad M_n(f(t),\ x) = (n+1) \sum_{v=0}^{n} p_{nv}(x) \int_0^1 p_{nv}(t)\ f(t)\ dt,$$

where $p_{nv}(x) = \binom{n}{v} x^v (1-x)^{n-v}$ for $0 \leq x \leq 1$ and $0 \leq v \leq n$. Derriennic [1] obtained interesting results concerning both uniform and L_p-approximation by (1.3).

The purpose of this paper is to determine degree of approximation in $L_p(I)$ by the linear combination (1.1) of operators (1.3). Therefore, for $f \in L_p(I)$, $1 \leq p < \infty$, our approximation method has the form:

$$L_n(f(t), k, x) = \sum_{j=0}^{k} c(j, k) M_{d_j n}(f(t), x), \tag{1.4}$$

where we write

$$M_n(f(t), x) = \int_0^1 H_n(x, t) f(t)\, dt,$$

with

$$H_n(x, t) = (n+1) \sum_{v=0}^{n} P_{nv}(x) P_{nv}(t). \tag{1.5}$$

We derive a global estimate in terms of the higher order integral modulus of smoothness of the function being approximated.

2. Degree of Approximation

Let $||\cdot||_p$ denote the p-norm in I below. We denote by $\omega_{2k+2}(f, p, \cdot)$, $k = 0, 1, 2, \ldots$, $1 \leq p < \infty$, the $2k+2$ order p-modulus of smoothness of f on I [8, p. 103].

THEOREM 2.1. *If $f \in L_p(I)$, $1 \leq p < \infty$, then for all sufficiently large n,*

$$||L_n(f, k, .) - f||_p \leq C_{p,k}(n^{-(k+1)} ||f||_p + \omega_{2k+2}(f, p, n^{-1/2}) \tag{2.1}$$

where the constant $C_{p,k}$ depends on k and p but is independent of f and n.

The method of proof is to first approximate in a smooth subspace of $L_p(I)$ (Lemma 2.5 below) and then use Peetre's K-functional to obtain the degree of approximation in $L_p(I)$. This approach is similar to that of Sinha in [7]. In particular, the proof of Lemma 2.5 below follows closely ideas developed in [7]. We require the following lemmas.

LEMMA 2.2. [1]. *For $x, t \in I$, $1 \leq p < \infty$, $j = 1, 2, \ldots$, and $n = 1, 2, \ldots$, we have*

$$\int_0^1 H_n(x, t)\, dt = \int_0^1 H_n(x, t)\, dx = 1, \tag{2.2}$$

$$||M_n(f)||_p \leq ||f||_p, \tag{2.3}$$

and

$$M_n((t-x)^{2j}, x) \leq C_1 n^{-j}, \tag{2.4}$$

where the constant C_1 depends on j but is independent of x and n.

LEMMA 2.3. *For every $r > 0$ and $x \in I$,*

$$M_n(|t-x|^r, x) \leq C_2 n^{-r/2}, \tag{2.5}$$

where the constant C_2 depends on r but is independent of x and n.

Proof. Choose a positive integer j such that $2j > r$. Using Hölder's inequality, (2.2) and (2.4) we have

$$\int_0^1 |t-x|^r H_n(x, t)\, dt \leq \left(\int_0^1 (t-x)^{2j} H_n(x, t) dt\right)^{r/2j} \left(\int_0^1 H_n(x, t) dt\right)^{1-r/2j} \leq C_2 n^{-r/2}$$

LEMMA 2.4. *For $v = 1, 2, \ldots, 2k+2$,*

$$\sup_{x \in I} |L_n((t-x)^v, k, x)| = 0(n^{-(k+1)}), \quad n \to \infty, \tag{2.6}$$

and

$$L_n(1, k, x) = 1, \text{ for } x \in I \text{ and } n = 1, 2, \ldots. \tag{2.7}$$

Proof. It follows from [1] that, for each $x \in I$ and each $v = 1, 2, \ldots$, $M_n((t-x)^v, x)$ can be expressed as a rational function in n. The degree of the numerator is less than the degree of the denominator and degree of both numerator and denominator depends on v. The denominator is independent of x and has distinct integer roots. The coefficients of the polynomial in n in the numerator are polynomials in x of degree at most v. Using partial fractions and (1.4) we obtain polynomials $a_s(x)$ of degree at most v and distinct integers α_s, $s = 1, \ldots, g(v)$, such that for $x \in I$, $v = 1, 2, \ldots, 2k+2$, and all n sufficiently large,

$$L_n((t-x)^v, k, x) = \sum_{j=0}^{k} c(j, k) \sum_{s=1}^{g(v)} \frac{a_s(x)}{d_j n - \alpha_s} =$$

$$= \sum_{s=1}^{g(v)} a_s(x) \sum_{r=0}^{\infty} \frac{\alpha_s^r}{n^{r+1}} \sum_{j=0}^{k} c(j, k)\, d_j^{-(r+1)}.$$

The result (2.6) now follows from the above, (1.2) and [5, p. 1228]. We obtain (2.7) from (1.4), (2.2), (1.2) and [5, p. 1228].

For $1 \leq p < \infty$ let $L_p^{(2k+2)}(I)$ denote the space of all functions $f \in L_p(I)$ such that the first $2k+1$ derivatives of f are absolutely continuous and $f^{(2k+2)} \in L_p(I)$. Our final lemma concerns approximation in the space $L_p^{(2k+2)}(I)$.

LEMMA 2.5. *If $p > 1$ and $f \in L_p^{(2k+2)}(I)$ then for all n sufficiently large,*

$$||L_n(f, k, .) - f||_p \leq C_3 n^{-(k+1)}(||f^{(2k+2)}||_p + ||f||_p), \tag{2.8}$$

where C_3 *is a constant which depends on* k *and* p *but is independent of* f *and* n.

If $f \in L_1(I)$, f *has* $2k+1$ *derivatives on* I *with* $f^{(2k)}$ *absolutely continuous on* I *and* $f^{(2k+1)}$ *of bounded variation on* I *then*

$$(2.9)\quad ||L_n(f, k, .) - f||_1 \leq C_3^1 n^{-(k+1)} (||f^{(2k+1)}||_{B.V.(I)} + ||f^{(2k+1)}||_1 + ||f||_1),$$

where C_3^1 *is a constant which depends on* k *but is independent of* f *and* n.

Proof. Assume $p > 1$. For $x \in I$ and $t \in I$, with the given assumptions on f, we can write

$$f(t) = \sum_{i=0}^{2k+1} \frac{(t-x)^i}{i!} f^{(i)}(x) + \frac{1}{(2k+1)!} \int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\, dw.$$

We have, using (2.7),

$$L_n(f, k, x) - f(x) = \sum_{i=1}^{2k+1} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{k} c(j, k) M_{a_j n}((t-x)^i, x) +$$

$$+ \frac{1}{(2k+1)!} \sum_{j=0}^{k} c(j, k) \int_0^1 H_{a_j n}(x, t) \int_x^t (t-w)^{2k+1} f^{(2k+2)}(w)\, dw\, dt = \Sigma_1 + \Sigma_2, \text{ say.}$$

It follows from (2.6) and [3, *p*. 5] that

$$(2.10)\quad ||\Sigma_1||_p \leq (\text{constant})\, n^{-(k+1)} \cdot (||f^{(2k+1)}||_p + ||f||_p).$$

In order to estimate Σ_2 we first estimate

$$\left\| M_n\left(\int_x^t |t-w|^{2k+1} \left| f^{(2k+2)}(w) \right| |dw|,\ x \right) \right\|_p = J, \text{ say.}$$

Let h_f denote the Hardy-Littlewood majorant [9, p. 244] of $f^{(2k+2)}$. Using Hölder's inequality and (2.4) we have

$$\int_0^1 H_n(x, t) \left| \int_x^t |t-w|^{2k+1} |f^{(2k+2)}(w)|\, dw \right| dt \leq \int_0^1 H_n(x, t)(t-x)^{2k+2} h_f(t)\, dt \leq$$

$$\leq \left(\int_0^1 (t-x)^{(2k+2)q} H_n(x, t)\, dt \right)^{1/q} \left(\int_0^1 |h_f(t)|^p H_n(x, t)\, dt \right)^{1/p} \leq$$

$$\leq (\text{constant})\, n^{-(k+1)} \left(\int_0^1 \int_0^1 |h_f(t)|^p H_n(x, t)\, dt\, dx \right)^{1/p}$$

It follows from Fubini's theorem, (2.2), and [9, *p*. 244] that

$$J \leq (\text{constant})\, n^{-(k+1)} \left(\int_0^1 \int_0^1 |h_f(t)|^p H_n(x, t)\, dt\, dx \right)^{1/p} \leq$$

$$(2.11)\quad \leq (\text{constant})\, n^{-(k+1)} \left(\int_0^1 |h_f(t)|^p\, dt \right)^{1/p} \leq (\text{constant})\, n^{-(k+1)}\, ||f^{(2k+2)}||_p.$$

Using (1.2), (1.4) and (2.11),

$$(2.12)\quad ||\Sigma_2||_p \leq (\text{constant})\, n^{-(k+1)}\, ||f^{(2k+2)}||_p.$$

Combining (2.10), (2.12) and using [3, *p*. 5], we obtain (2.8). Note that the constants on the right hand side of estimates (2.10) and (2.12) depend on k and p.

Now assume $p = 1$ and χ is the characteristic function of I. With the given assumptions on f we can write, for almost all $x \in I$ and all $t \in I$,

$$f(t) = \sum_{i=0}^{2k+1} \frac{(t-x)^i}{i!} f^{(i)}(x) + \frac{1}{(2k+1)!} \int_x^t (t-w)^{2k+1}\, df^{(2k+1)}(w).$$

The proof of (2.9) proceeds exactly as that of (2.8), except that we estimate

$$\Sigma_2^1 = \frac{1}{(2k+1)!} \sum_{j=0}^{k} c(j, k) \int_0^1 H_{a_j n}(x, t)\, \chi(t) \int_x^t (t-w)^{2k+1}\, df^{(2k+1)}(w)\, dt$$

as follows.

First consider

$$\left\| M_n\left(\chi(t) \left| \int_x^t |t-w|^{2k+1} \,|df^{(2k+1)}(w)|,\ x \right. \right) \right\|_1 = J^1, \text{ say.}$$

For each n let $r = r(n) = [n^{1/2}]$, so that $(r+1)n^{-1/2} \geq 1$. We have, since χ is the characteristic function of $[0, 1]$,

$$J^1 = \int_0^1 \int_0^1 \chi(t)\, H_n(x, t) \left| \int_x^t \chi(w) |t-w|^{2k+1} \right| |df^{(2k+1)}(w)|\, dt\, dx \leq$$

$$\leq \sum_{l=0}^{r} \int_0^1 \left\{ \int_{x+l n^{-1/2}}^{x+(l+1)n^{-1/2}} \chi(t)\, H_n(x, t) |t-x|^{2k+1} \cdot \left(\int_x^{x+(l+1)n^{-1/2}} \chi(w) |df^{(2k+1)}(w)| \right) dt + \right.$$

$$+\int\limits_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}} \chi(t)\, H_n(x,\,t)|t-x|^{2k+1}\cdot\left(\int\limits_{x-(l+1)n^{-1/2}}^{x}\chi(w)\,|df^{(2k+1)}(w)|\right)dt\Bigg\}\,dx.$$

Let $\chi_{x,c,d}(w)$ denote the characteristic function of the interval $[x-cn^{-1/2},\, x+dn^{-1/2}]$ where c, d are non-negative integers. Then we have

$$J^1 \le \sum_{l=1}^{r}\int_0^1\Bigg\{\int\limits_{x+ln^{-1/2}}^{x+(l+1)n^{-1/2}}\chi(t)H_n(x,\,t)l^{-4}\,n^2|t-x|^{2k+5}$$

$$\cdot\left(\int\limits_{x}^{x+(l+1)n^{-1/2}}\chi(w)\chi_{x,0,l+1}(w)\,|\,df^{(2k+1)}(w)\,|\right)dt$$

$$+\int\limits_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}}\chi(t)H_n(x,\,t)l^{-4}\,n^2|t-x|^{2k+5}$$

$$\cdot\left(\int\limits_{x-(l+1)n^{-1/2}}^{x}\chi(w)\;\chi_{x,l+1,0}(w)\,|\,df^{(2k+1)}(w)\,|\right)dt\Bigg\}\,dx$$

$$+\int_0^1\int\limits_{-n^{-1/2}}^{1+n^{-1/2}}\chi(t)H_n(x,\,t)\,|\,t-x\,|^{2k+1}$$

$$\cdot\left(\int\limits_{x-n^{-1/2}}^{x+n^{-1/2}}\chi(w)\;\chi_{x,1,1}(w)\,|\,df^{(2k+1)}(w)\,|\right)dt\,dx$$

$$\le \sum_{l=1}^{r}\Bigg\{l^{-4}\,n^2\int_0^1\left(\int\limits_{x+ln^{-1/2}}^{x+(l+1)n^{-1/2}}\chi(t)H_n(x,\,t)\,|\,t-x\,|^{2k+5}\right.$$

$$\cdot\left(\int_0^1\chi_{x,0,l+1}(w)\,|\,df^{(2k+1)}(w)\,|\right)dt$$

$$+\int\limits_{x-(l+1)n^{-1/2}}^{x-ln^{-1/2}}\chi(t)H_n(x,\,t)|t-x|^{2k+5}\cdot\int_0^1\chi_{x,l+1,0}(w)|df^{(2k+1)}(w)|\,dt\Bigg)\,dx\Bigg\}$$

$$+\int_0^1\int\limits_{-n^{-1/2}}^{1+n^{-1/2}}\chi(t)H_n(x,\,t)|t-x|^{2k+1}\cdot\left(\int_0^1\chi_{x,1,1}(w)|df^{(2k+1)}(w)\,|\right)dt\,dx.$$

We use (2.5) and Fubini's theorem to obtain

$$J^1 \le (\text{constant})\; n^{-(2k+1)/2}\Bigg\{\sum_{l=1}^{r}l^{-4}\left(\int_0^1\int_0^1\chi_{x,0,l+1}(w)\,|\,df^{(2k+1)}(w)\,|\,dx\;+\right.$$

$$\left.+\int_0^1\int_0^1\chi_{x,l+1,0}(w)\,|\,df^{(2k+1)}(w)\,|\,dx\right)+\int_0^1\int_0^1\chi_{x,1,1}(w)\,|\,df^{(2k+1)}(x)\,|\,dx\Bigg\}$$

$$=(\text{constant})\; n^{-(2k+1)/2}\Bigg\{\sum_{l=1}^{r}l^{-4}\left(\int_0^1\left(\int_0^1\chi_{x,0,l+1}(w)\,dx\right)|\,df^{(2k+1)}(w)\,|\right.$$

$$\left.+\int_0^1\left(\int_0^1\chi_{x,l+1,0}(w)\,dx\right)|df^{(2k+1)}(w)|\right)+\int_0^1\left(\int_0^1\chi_{x,1,1}(w)dx\right)|\,df^{(2k+1)}(w)\,|\Bigg\}$$

$$\le(\text{constant})\; n^{-(2k+1)/2}\Bigg\{\sum_{l=1}^{r}l^{-4}\left(\int_0^1\left(\int\limits_{w-(l+1)n^{-1/2}}^{w}dx\right)|\,df^{(2k+1)}(w)\,|\right.$$

$$\left.+\int_0^1\left(\int\limits_{w}^{w+(l+1)n^{-1/2}}dx\right)|df^{(2k+1)}(w)|\right)+\int_0^1\left(\int\limits_{w-n^{-1/2}}^{w+n^{-1/2}}dx\right)|df^{(2k+1)}(w)|\Bigg\}$$

$$\le(\text{constant})\; n^{-(k+1)}\,||f^{(2k+1)}||_{B.V.(I)}. \tag{2.13}$$

Using (1.2), (1.4) and (2.13) we have

$$||\Sigma_2^1||_1 \le (\text{constant})\; n^{-(k+1)}||f^{(2k+1)}||_{B.V.(I)}, \tag{2.14}$$

where the constant on the right hand side of (2.14) depends on k.

Combining (2.14) with the analog to (2.10) for $p=1$ completes the proof of (2.9).

Proof of Theorem 2.1. Let $1\le p<\infty$, $f\in L_p(I)$ and $g\in L_p^{(2k+2)}(I)$. It follows from (1.2), (1.4) and (2.3) that $\{L_n\}$ is a uniformly bounded sequence of linear operators on $L_p(I)$. Let $R_{p,k}>0$ be a uniform bound for $\{L_n\}$ and apply Lemma 2.5 and [3, p. 5] to obtain

$$||L_n(f,\,k,\,\cdot)-f||_p \le (1+R_{p,k})||f-g||_p + T_{p,k}\,n^{-(k+1)}\,(||g^{(2k+2)}||_p+||g||_p)$$

for all n sufficiently large, where $T_{p,k}$ is a constant independent of f, g and n. Take the infimum over all $g\in L_p^{(2k+2)}(I)$ and use Peetre's K-functional [4, p. 300] to obtain (2.1).

The choice $k=0$ in (1.4) and (2.1) yields an estimate for approximation in $L_p(I)$ by the operators (1.3) (see also [1]).

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