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THE GENERALIZED SOLUTION OF A NONLINEAR DEGENERATE PARABOLIC EQUATION AND ITS NUMERICAL COMPUTATION

by

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Our aim is to solve numerically the problem (1.1)—(1.3) using the classical implicit difference scheme and prove with the aid of the numerical solution that it has a generalized solution. This is done by means of a continuous extension of the numerical solution which we show that it converges in L^p to the exact one (Theorem 5.2).

One dimensional problems of the same type were studied in [3], [10]. The explicit difference scheme was used in the same purpose by the author in [11—14] but under more restrictive conditions on φ, u_0, u_1, namely φ convex, u_0 of class C^2 and u_1 independent of t [14, § 3]. A special attention is devoted in § 3, to the problem of fulfilment of the nonhomogeneous boundary and initial conditions.

1. The differential problem

The problem we are concerned with can be written as:

$$(1.1)\qquad \frac{\partial u}{\partial t} = \Delta\varphi(u) + a(x, y, t) \quad \text{on} \quad Q = \Omega \times]0, T[$$

$$(1.2)\qquad u(x, y, 0) = u_0(x, y) \qquad (x, y) \in \Omega$$

$$(1.3)\qquad u(x, y, t)|_S = u_1(x, y, t) \qquad S = \partial\Omega \times [0, T],$$

where $\Omega \in \mathbf{R}^2$ is a bounded, convex domain, $0 < T < +\infty$. We consider two spatial variables only, for the sake of simplicity.

The functions occuring in the above problem will be subject to the following assumption:

(*A*)
(*i*) $u_0 \in C(\overline{\Omega})$; $u_1 \in C(S)$; $a \in C(\overline{Q})$; $u_0, u_1, a \geq 0$

(*ii*) $\varphi \in C^1(\mathbf{R}_+)$, $\varphi(u)$ and $\varphi'(u) > 0$ for $u > 0$, $\varphi(0) = 0$.

Denote $M \geq u_0, u_1, a$.

We notice that for the examples we have in mind (such as $\varphi(u) = u^m$, $m < 1$), $\varphi'(0) = 0$. However this condition is not necessary in our considerations.

DEFINITION. A function $u \in L^\infty(Q)$, $u \geq 0$ (a.e.), is said to be a weak solution of (1.1) — (1.3) if:

(i) It satisfies (1.2), (1.3) in a generalized sense.

(ii) $\varphi(u) \in L^2(0, T; H^1(\Omega))$.

(iii) For any $f \in H^1(Q)$ such that $f|_S = 0$,

$$\int_Q \left(u\frac{\partial f}{\partial t} - \frac{\partial\varphi(u)}{\partial x}\frac{\partial f}{\partial x} - \frac{\partial\varphi(u)}{\partial y}\frac{\partial f}{\partial y}\right) dx\,dy\,dt + \int_Q af\,dx\,dy\,dt + \tag{1.4}$$
$$+ \int_\Omega u_0(x, y) f(x, y, 0)\,dx\,dy.$$

$$S_1 = S \cup \{(x, y, T); (x, y) \in \overline{\Omega}\}.$$

Condition (*i*) is to be interpreted in the following sense:

There exists a sequence $v_n \in C^\infty(\overline{Q})$ such that, $v_n \to u$ in $L^2(Q)$, $v_n|_S \to u_1$ in $L^2(S)$ and $v_n(x, y, 0) \to u_0(x, y)$ in $L^2(\Omega)$.

THEOREM 1.1. *Under the Assumption (A), the problem* (1.1) — (1.3) *has at most one weak solution.*

Proof: (see [9]) Suppose that there are two solutions u_1, u_2 and prove that they coincide a.e. Then by (1.4),

(1.5)
$$\int_Q \left[(u_1 - u_2)\frac{\partial f}{\partial t} - \frac{\partial}{\partial x}(\varphi(u_1) - \varphi(u_2))\frac{\partial f}{\partial x} - \frac{\partial}{\partial y}(\varphi(u_1) - \varphi(u_2))\frac{\partial f}{\partial y}\right] dx\,dy\,dt = 0.$$

The particular function

$$f(x, y, t) = \int_T^t [\varphi(u_1) - \varphi(u_2)]d\tau$$

has the required properties for (1.4). Replacing it in (1.5) we get:

$$\int_Q (u_1 - u_2)(\varphi(u_1) - \varphi(u_2))\,dx\,dy\,dt + \int_Q \left\{\left[\int_t^T \frac{\partial}{\partial x}(\varphi(u_1) - \varphi(u_2))\,d\tau\right] \times\right.$$
$$\left.\times \frac{\partial}{\partial x}(\varphi(u_1) - \varphi(u_2)) + \int_t^T \left[\frac{\partial}{\partial y}(\varphi(u_1) - \varphi(u_2))\,d\tau\right]\frac{\partial}{\partial y}(\varphi(u_1) - \varphi(u_2))\right\} dx\,dy\,dt.$$

The second integral can be written as:

$$-\frac{1}{2}\int_Q \frac{\partial}{\partial t}\left\{\left(\int_t^T \left[\frac{\partial\varphi(u_1)}{\partial x} - \frac{\partial\varphi(u_2)}{\partial x}\right] d\tau\right)^2 + \left(\int_t^T \left[\frac{\partial\varphi(u_1)}{\partial y} - \frac{\partial\varphi(u_2)}{\partial y}\right] d\tau\right)^2\right\} dx\,dy\,dt =$$
$$= \frac{1}{2}\int_\Omega \left\{\left(\int_0^T \left(\frac{\partial\varphi(u_1)}{\partial x} - \frac{\partial\varphi(u_2)}{\partial x}\right) dt\right)^2 + \left(\int_0^T \left(\frac{\partial\varphi(u_1)}{\partial y} - \frac{\partial\varphi(u_2)}{\partial y}\right) dt\right)^2\right\} dx\,dy$$

which is positive.

Hence

$$\int_Q (u_1 - u_2)(\varphi(u_1) - \varphi(u_2))\,dx\,dy\,dt = 0$$

so that $u_1 = u_2$ a.e. on Q.

Remark 1.1. Our interpretation of condition (*i*) of the Definition, implies the following assertion: If the corresponding traces exist they coincide with the data functions. As it was proved in [7, Ch. VII, Th. 2.1] the traces exist provided that Ω is sufficiently regular.

2. The difference problem

Consider the rectangular mesh R_h with step $h > 0$ in the Ox, Oy directions and $\tau > 0$ in the Ot direction, such that $0 \in R_h$. Denote $\overline{\Omega}_h = R_h \cap \overline{\Omega}$, $\overline{Q}_h = \overline{Q} \cap R_h$ and $x_i = ih$, $y_j = jh$, $t_k = k\tau$; $U_{ij}(k) = U(x_i, y_j, t_k) = U(k)$.

Put

$$U_{\bar{t}}(k) = \frac{U(k) - U(k-1)}{\tau}$$

and similarly $U_{\bar{x}}$, $U_{\bar{y}}$ for the backward differences in the space variables. The forward differences will be denoted by U_x, U_y, U_t.

As usual

$$\Delta_h U(k) = U_{\bar{x}x}(k) + U_{\bar{y}y}(k).$$

To the differential problem (1.1) — (1.3) we associate the following difference scheme:

$$\text{(2.1)}\quad U_{\bar{t}}(k) = \Delta_h\varphi(U(k)) + a(k) \text{ on } Q_h$$

$$\text{(2.2)}\quad U(0) = u_{0h}$$

$$\text{(2.3)}\quad U|_{\Gamma_h} = u_2(x, y, k\tau) \quad k = 1, 2, \ldots K = \left[\frac{T}{\tau}\right].$$

Here

$$u_{0h} = u_0|_{\overline{\Omega}_h}$$

and Γ_h is the set of points $(x_i, y_j) \in R_h$ such that at least one of the four neighbours: (x_{i+1}, y_j), (x_{i-1}, y_j), (x_i, y_{j+1}), $(x_i y_{j-1})$ lies outside $\overline{\Omega} \cdot \Omega_h = \overline{\Omega}_h \setminus \Gamma_h$,

$$Q_h = \{(x, y, t)\,;\ (x, y) \in \Omega_h,\ t = t_1, t_2, \ldots, t_K\}.$$

The functions u_2 and u_{0h} are defined as follows:

$$u_2(x, y, t) = u_1(x^*, y^*, t)$$

where $(x^*, y^*) \in \partial\Omega$ is the nearest point to (x, y) (or one of them but the same on all levels).

THEOREM 2.1. *If assumption (A) holds and U is a solution of the difference problem* (2.1) — (2.3), *then*

$$0 \leq U \leq M_0,$$

where $M_0 = (1 + T)M$.

Proof: We show that $U(k) \leq (1 + k\tau)M$ for $k = 1, 2, \ldots, K$. Assume the contrary. Then there exists a triplet of indices (m, n, k), $k > 1$ so that $U_{mn}(k) > (1 + k\tau)M$ and $U_{ij}(k - 1) \leq (1 + (k - 1)\tau)M$ for all (i, j); $U_{mn}(k)$ being maximum on $\overline{\Omega}_h$, By the definition of M,

$$(x_m, y_n, \tau k) \in \Omega_h \quad\text{and}\quad \Delta_h U_{mn} \leq 0.$$

But

$$U_{mn}(k) = U_{mn}(k - 1) + \tau\Delta_h\varphi(U_{mn}(k)) + a(k)$$

and

$$U_{mn}(k) \leq (1 + k\tau)M,$$

which leads to a contradiction.

Now consider the following linear homogeneous algebraic system:

$$\text{(2.4)}\quad \begin{aligned} &V_{ij} = \tau\Delta_h(\alpha_{ij}V_{ij}) \quad (x_i, y_j) \in \Omega_h \\ &V_{ij}|_{\Gamma_h} = 0 \quad \alpha_{ij} \geq 0. \end{aligned}$$

LEMMA 2.1. *For any given* α_{ij}, *the system* (2.4) *admits only the trivial solution.*

Proof: If, on the contrary, for an $(x_m, y_n) \in \Omega_h$, e.g. $V_{mn} < 0$, this couple can be chosen so that $\Delta_h(\alpha_{mn}V_{mn}) \geq 0$, but this contradicts (2.4). A similar argument applies when $V_{mn} > 0$.

Consider now, the linear system:

$$\text{(2.5)}\quad \begin{aligned} &W_{ij}(k) = Z_{ij}(k) + \tau\Delta_h(\alpha_{ij}(k)W_{ij}(k)) + \tau a_{ij}(k) \quad (x_i, y_j) \in \Omega_h \\ &W_{ij}(k) = Z_{ij}(k) \text{ on } \Gamma_h \\ &\alpha_{ij} \geq 0, \quad Z : \overline{\Omega}_h \to \mathbf{R}. \end{aligned}$$

COROLLARY 2.1. *For a given Z and* $k = 1, 2, \ldots, K$ *fixed,* (2.5) *has a unique solution.*

This is an immediate consequence of Lemma 2.1.

Another consequence of the same Lemma refers to the system:

$$\text{(2.6)}\quad \begin{aligned} &W_{ij}(k) = W_{ij}(k - 1) + \tau\Delta_h(\alpha_{ij}(k)W_{ij}(k)) + \tau a_{ij}(k) \text{ on } \Omega_h \\ &W_{ij}(k) = u_2(x_i, y_j, t_k) \quad (x_i, y_j) \in \Gamma_h,\ k = 1, 2, \ldots, K. \\ &W_{ij}(0) = u_0(x_i, y_j) \quad \text{on } \overline{\Omega}_h. \end{aligned}$$

Here $V : \overline{Q}_h \to \mathbf{R}$, $V \geq 0$, $\varphi(0) = 0$, $\varphi \in C^1(\mathbf{R}_+)$,

$$\alpha_{ij}(k) = \begin{cases} \dfrac{\varphi(V_{ij}(k))}{V_{ij}(k)} & V_{ij}(k) \neq 0 \\ \varphi'(0) & V_{ij}(k) = 0. \end{cases}$$

LEMMA 2.2. *Suppose condition (A) holds and* $V : \overline{Q}_h \to \mathbf{R}$ *is given such that* $V \geq 0$ *and* $V|_{\Gamma_h} = u_2$. *Then* (2.6) *has a unique solution*

$$W : \overline{Q}_h \to \mathbf{R}$$

such that $W \geq 0$.

Proof: It is readily seen from Corollary 2.1 that there is a unique solution for the system. If W takes negative values, there exists a couple (m, n) such that

$$W_{mn}(k) < 0, \qquad \Delta_h(\alpha_{mn}W_{mn}(k)) \geq 0$$

and $W_{mn}(k - 1) \geq 0$, but this is impossible according to the equation. The lemma is proved.

For a fixed $k \in \{1, 2, \dots, K\}$, (2.6) defines on operator:

$$G(k): \mathbf{R}^s_+ \to \mathbf{R}^s_+, \; V(k) \to W(k),$$

$s = \operatorname{card} \Omega_h$.

LEMMA 2.3. *Under the assumption (A) the operator $G(k)$ has a unique fixed point for any $k = 1, 2, \dots, K$.*

Proof: Since $W \geqq 0$,

$$\|W(k)\|_1 = h^2 \sum_{\Omega_h} |W_{ij}(k)| = h^2 \sum_{\Omega_h} W_{ij}(k) =$$

$$= h^2 \sum_{\Omega_h} (W_{ij}(k-1) + \tau\Delta_h(\alpha_{ij}(k) W_{ij}(k)) + a_{ij}(k).$$

Hence, if we put $\lambda = \tau/h^2$,

$$\|W(k)\|_1 \leq h^2 \sum_{\Omega_h} W_{ij}(k-1) + 2\lambda h^2 \sum_{\Gamma_h} \varphi(W_{ij}(k)) + \tau h^2 \sum_{\Omega_h} a_{ij}(k),$$

because $W = V$ on Γ_h.

Denote

$$\rho(k) = h^2 \sum_{\Omega_h} W_{ij}(k-1) + 2h\,\lambda\, m(\partial\Omega) + \tau\, M\, m(\Omega)$$

and

$$S^+_\rho(k) = \{x \in \mathbf{R}^s;\; x \geqq 0,\; \|x\|_1 \leqq \rho(k)\}.$$

Thus

$$G(k): \mathbf{R}^s_+ \to S^+_\rho(k)$$

and Brower's theorem is applicable. The uniquness follows from Lemma 2.2.

Remark 2.1. The fixed points $U(k)$, $k = 1, 2, \dots, K$, satisfy:

(2.7) $$U_{\bar t}(k) = \Delta_h \varphi(U(k)) + a(k) \quad \text{on} \quad Q_h$$

(2.8) $$U(0) = u_{0h}$$

(2.9) $$U|_{\Gamma_h} = u_2.$$

3. Mesh-functions

To begin with, we recall the formula of partial summation.

Suppose U, V are vectors with components

$$U_i,\; V_i,\; p \leqq i \leqq q \qquad p, q \in Z,\; p < q;$$

(3.1) $$h \sum_{k=p}^{q-1} (V_x)_k\, U_k = -h \sum_{k=p+1}^{q} V_k (U_{\bar x})_k + U_q V_q - U_p V_p.$$

In order to define extensions of mesh-function, we introduce the "cells";

$$\omega_{ij} = \{(x, y) \in \mathbf{R}^2;\; ih < x < (i+1)h,\; jh < y < (j+1)h\}$$

$$q_{ij}(k) = \omega_{ij} \times \,]k\tau, (k+1)\tau[.$$

Denote

$$\underline{\Omega}_h = \bigcup_{\omega_{ij} \subset \Omega} \bar\omega_{ij}, \qquad \underline{Q}_h = \bigcup_{q_{ij} \subset Q} \bar q_{ij}.$$

We observe that $\underline{\Omega}_h$ is the rectangular domain generated by the mesh-points also denoted by $\bar\Omega_h$. This is the greatest such domain contained in $\bar\Omega$. $\underline{Q}_h$ has a similar property. In the same way the smallest rectangular domain containing $\bar\Omega$ respectively $\bar Q$ are:

$$\bar\Omega^*_h = \bigcup_{\omega_{ij} \cap \Omega \neq \emptyset} \bar\omega_{ij}, \qquad \bar Q_h = \bigcup_{q_{ij} \cap Q \neq \emptyset} \bar q_{ij}.$$

In the sequel we shall use three types of finite-element interpolants of mesh-functions.

a) (0)-interpolation: Given $V_h : \bar Q_h \to \mathbf{R}$. Its (0)-interpolate $\tilde V_h$ is defined in the rectangular domain $\underline{Q}_h$ as follows:

$$\tilde V_h|_{q_{ij}(k)} = V_{ij}(k)$$

for any $q_{ij}(k) \subset Q_h$.

b) (1)-interpolation: This assigns to V_h a continuous function V'_h defined on $\underline{Q}_h$, such that on each cell $q_{ij}(k)$ it is the Lagrange interpolate of degree one in each variable, of the values of V on the vertices. Clearly, V'_h has integrable first order generalized derivatives.

c) Mixed interpolation: $V_{(1)}$ is defined on the rectangular domain $\underline{Q}_h$ in the following manner: On each $q_{ij}(k)$ it is linear in y and t and constant in x. $V_{(1)}(x_i, y, t)$ is an interpolation polinomial on the face $y_j \leqq y \leqq y_{j+1}$, $t_k \leqq t \leqq t_{k+1}$. Analogously one defines $V_{(2)}$ for constant y. Clearly, all the above extensions can be adapted when $\underline{Q}_h$ is changed into $\underline{Q}^-_h$ or $\underline{\Omega}_h$. In what follows extensions defined on $\underline{Q}_h$ or $\underline{Q}^-_h$ will automatically be prolonged for $t \in [K\tau, T]$ by

$$V'(t) = \tilde V(t) = V(K\tau), \qquad t \in [K\tau, T].$$

Thus the qualities of V' and $\tilde V$ will remain unchanged.

It is easily seen that:

(3.2) $$\frac{\partial V'}{\partial x} = (V_x)_{(1)} \quad \text{on any} \quad q_{ij}(k)$$

and analogously for y.

Because we are mainly concerned with the situation when $h, \tau \to 0$ we shall deal with (generalized) sequences of mesh functions. Neverteless, we shall speak of a function V (or V_h) defined on R_h.
Next, we recall an important theorem due to Ladyzhenskaia [5].

THEOREM 3.1. *Suppose that:*

(*i*) *There is a constant C independent of h and τ such that for $V: \overline{Q}_h \to \mathbf{R}$:*

$$\tau h^2 \sum_{Q_h} V^2 \leq C, \tag{3.3}$$

(*ii*) *V is defined on the mesh-points outside $\overline{Q}_h$ so that (3.3) holds on the whole R_h.*

Then, if one of the sequence $\{\widetilde{V}\}$, $\{V'\}$, $\{V_{(1)}\}$, $\{V_{(2)}\}$ is weakly convergent in $L^2(Q)$ when $h, \tau \to 0$, the same is true for the other three sequences of extensions.

The properties we are formulating below will play an important rôle in the following sections.
(P): The couple (Ω, u_1) is said to have the property (P) if there exists a function $f: \overline{Q} \to \mathbf{R}$ such that:

(*i*) $f|_S = u_1 \qquad S = \partial\Omega \times [0, T]$

(*ii*) $f \in C(\overline{Q})$ with bounded derivatives up to the second order in x, y and first order in t.

(P_h): The couple (Ω, u_1) is said to have the property (P_h) if there exists a mesh-function $f_h: \overline{Q}_h \to \mathbf{R}$ such that

(*i*) $f_h(k)|_{\Gamma_h} = u_2 \qquad k = 1, 2, \ldots, K$

(*ii*) $f_{\bar{t}}(k), f_x(k), f_y(k), f_{x\bar{x}}(k), f_{y\bar{y}}(k)$ are bounded on $\overline{\Omega}_h$, $k = 1, 2, \ldots, K$.

The differences are taken on points of the mesh $\overline{Q}_h$ on which they make sense. u_2 was defined for condition (2.3).

Remark 3.1. If (P) holds, the restriction of f to the mesh $\overline{Q}_h$ satisfies condition (*ii*) of (P_h) and

$$f(k)|_{\Gamma_h} = u_2 + r_h, \quad k = 1, 2, \ldots, K, \quad |r_h| < Ch,$$

C independent of h, τ.
Thus (P_h) is "nearly" satisfied.

Remark 3.2. If u_1 is independent of x, y, then $u_2 = u_1$ and both (P) and (P_h) hold.

This is a situation very often occuring in practice. Many authors have studied the problem of the conditions under which (P) holds. In connection with parabolic problems condition (P) appears in Friedman [4, Ch. III, § 4] (Class $C_{2+\alpha}$) and Ladyzhenskaia [6] (Class $O^{2,1}$).

4. First-order differences

In this section we are concerned with first order differences of the discrete solution U and their boundedness.
First here is some notation:

$$\Omega_h^+ = \{(x_i, y_j) \in \Omega_h;\ (x_{i+1}, y_j) \text{ and } (x_i, y_{j+1}) \in \overline{\Omega}_h\}$$

$$Q_h^+ = \{(x_i, y_j, t_k) \in R_h;\ (x_i, y_j) \in \Omega_h^+,\ k = 1, 2, \ldots, K\}.$$

$$\Gamma_h^+ = \{(x_i, y_j) \in \Omega_h;\ (x_{i-1}, y_j) \in \Gamma_h\};\ \Gamma_h^- = \{(x_i, y_j) \in \Gamma_h;\ (x_{i-1}, y_j) \in R_h.$$

${}^+\Gamma_h$ and ${}^-\Gamma_h$ have similar meanings regarding y.
In the sequel we shall always consider the solution U of (2.1) — (2.3) extended over the mesh-points of $\overset{*}{\overline{Q}}_h$, in the following way:

$$U(x_i, y_j, t_k) = U(x^*, y^*, t_k), \quad (x_i, y_j) \in \overline{\Omega}_h^* \setminus \overline{\Omega}_h$$

where $(x^*, y^*) \in \partial\Omega$ is the nearest point to (x_i, y_j).

THEOREM 4.1. *Suppose that*

(*i*) *U is the solution of* (2.1) — (2.3)

(*ii*) *(Ω, u_1) has the property (P_h)*

(*iii*) *Condition (A) holds*

(*iv*) *u_1 is Lipschitz continuous.*

Then there exists a constant C independent of h, τ such that

$$\tau h^2 \sum_{Q_h^{*+}} (\varphi_x(U)^2 + \varphi_y(U)^2) < C$$

Proof: Condition (*ii*) ensures the existence of a function $V: \overline{Q}_h \to \mathbf{R}$ such that $V|_{\Gamma_h} = u_2$ with bounded differences up to the second order in x, y and first order in t. We take $V = U$ outside $\overline{Q}_h$. Put $W = U - V$} Multiplying both sides of (2.1) by $\tau h^2 W(k)$ and summing up, we get

$$\tau h^2 \sum_{\Omega_h} W(k) U_{\bar{t}}(k) = h^2 \sum W(k) [\Delta_h \varphi(U(k)) + a(k)] \qquad k = 1, 2, \ldots, K. \tag{4.1}$$

The left-hand side can be transformed using the identity:

$$a(a - b) = \frac{1}{2} [a^2 - b^2 + (a - b)^2]$$

into

$$\frac{1}{2} h^2 \sum_{\Omega_h} (U^2(k) - U^2(k-1) + \tau^2 U_{\bar{t}}^2(k)) - \tau h^2 \sum_{\Omega_h} V(k)\, U_t(k-1). \tag{4.2}$$

Since $W = 0$ on $\bar{Q}_h^* \setminus Q_h$ and $W_x = 0$ on Γ_h^- ; $W_y = 0$ on $^-\Gamma_h$,

$$\tau h^2 \sum_{\Omega_h} W(k)\Delta_h \varphi(U(k)) = \tau h^2 \sum_{\bar{\Omega}_h} W(k)\Delta_h \varphi(U(k)) = \tag{4.3}$$
$$= -\tau h^2 \sum_{\Omega_h^+} [W_x(k)\, \varphi_x(U(k)) + W_y(k)\, \varphi_y(U(k))].$$

Further we have

$$-h^2 \sum_{\Omega_h^+} V_x\, \varphi_x(U) = h^2 \sum_{\Omega_h} V_{x\bar{x}}(U) + h \sum_{\Gamma_h^+} V_{\bar{x}}\varphi(U) - h \sum_{\Gamma_h^-} V_{\bar{x}}\varphi(U).$$

A similar identity is valid for

$$-h^2 \sum_{\Omega_h^+} V_y\, \varphi_y(U).$$

Replacing (4.2), (4.3), (4.4) in (4.1) and summing up for $k = 1, 2, \ldots, K$, we get after applying once more (3.1) :

$$\tau h^2 \sum_{Q_h^+} (U_x\, \varphi_x(U) + U_y\, \varphi_y(U)) \le \tau h^2 \sum_{Q_h} |\Delta_h U|\varphi(U) + \tau h \sum_{k=1}^{h} \Big(\sum_{\Gamma_h^+} V_{\bar{x}}\varphi(U) +$$
$$+ \sum_{^+\Gamma_h} V_{\bar{y}}\varphi(U) - \sum_{\Gamma_h^-} V_{\bar{x}}\varphi(U) - \sum_{^-\Gamma_h} V_{\bar{y}}\varphi(U)\Big) + \tau h^2 \sum_{Q_h} |W| a +$$
$$+ \tau h^2 \sum_{\bar{Q}_h} |V_{\bar{t}}| U + h^2 \sum_{\bar{\Omega}_h} (U(K)V(K) - V(0)U(0)).$$

Taking into account (P_h) and Theorem 2.1 it follows that there exists a constant C independent of the mesh sizes such that

$$\tau h^2 \sum_{Q_h^+} (U_x\, \varphi_x(U) + U_y\, \varphi_y(U)) \le C.$$

Finally we notice that $\varphi_x(U)^2 \leq \varphi'(M_0)\, \varphi_x(U) U_x$ and that $|\varphi_x(U)|$ is bounded on $Q_h^{*+} \setminus Q_h^+$, so our estimate is true.

Remark 4.1. It is readily seen from the proof of the above theorem that in Property (P_h) it is enough to suppose instead of the boundedness of the second order differences, that of Δ_h in the L^1 discrete norm.

Remark 4.2. Theorem 4.1 holds when (ii) is replaced by

(ii') (Ω, u_1) has the property (P).

Indeed, let's take in this case $V = f|_{Q_h}$. Then the right hand side of (4.3) is to be replaced by

$$-\tau h^2 \sum_{\Omega_h^-} [W_x(k)\varphi_x(U(k)) + W_y(k)\varphi_y(U(k))] - \tau h^2 \sum_{\Gamma_h^-} W(k)\varphi_x(U(k)) -$$
$$-\tau h^2 \sum_{^-\Gamma_h} W(k)\, \varphi_y(U(k)) + \tau h \sum_{\Gamma_h^-} W(k)\, \varphi_x(U(k)) + \tau h \sum_{^-\Gamma} W(k)\, \varphi_h(U(k)).$$

Recalling now that according to Remark 3.1, there exists a constant C independent of h, such that :

$$|W(k)| \leqq Ch, \text{ on } \Gamma,\ |W_x(k)| \leqq C \text{ on } \Gamma_h^-,\ |W_y(k)| \leqq C \text{ on } \Gamma_h^+,$$

we have for C independent of h,

$$h \sum_{\Gamma^-} (h|W_x(k)\varphi_x(V(k))| + |W(k)||\varphi_x(U(k))|) < C$$

$$h \sum_{^-\Gamma} (h|W_y(k)\varphi_y(U(k))| + |W(k)|\, |\varphi_y(U(k))|) < C.$$

Thus, the proof of Theorem 4.1 can proceed unchanged. Before passing to the next theorem we introduce the following notation :

$$M = \max \{i\,;\ \exists y_j : (x_i, y_j) \in \bar{\Omega}_h\}\,;\ m = \min \{i\,;\ \exists y_j : (x_i, y_j) \in \bar{\Omega}_h\}$$

and similarly N, n for y.
For a given

$$n \leqq j \leqq N,\ M(j) = \max \{i\,;\ (x_i, y_j) \in \Omega_h\}$$

The meaning of $m(j)$, $N(i)$, $n(i)$ is clear.

THEOREM 4.2. *Suppose that* (A) *holds and* U *is the solution of problem* (2.1) — (2.3). *Assume that* u_1 *is Lipschitz continuous in* x, y, t. *Let condition* (P) *or* (P_h) *hold. Then there exits a constant* C *independent of* h, *such that*

$$\max_{[0,\,T]} \left\| \frac{\partial}{\partial t} U'_h \right\|_{H^{-3}(\Omega)} < C.$$

Proof: We have to show that

$$\left| \int_\Omega \frac{\partial}{\partial t} U'(x, y, t)(x, y)\,dx\,dy \right| < C||\psi||_{H^3(\Omega)},$$

for any $\psi \in C_0^\infty(\Omega)$ and $t \in [0, T]$. Recall that U' was extended to $\bar{\Omega}_h^* \times [0, T]$.

For $(x, y, t) \in \bar{q}_{ij}(k)$

$$U'(x, y, t) = \frac{1}{\tau h^2} \Sigma\, U_{ij}(k)\, L_{ijk}(x, y, t)$$

where the sum is extended over the vertices of $q_{ij}(k)$ and for $P \in R_h \cap \cap\, \bar{q}_{ij}(k)$:

$$L_{ijk}(P) = \begin{cases} 1 & \text{for } \ P = (x_i, y_j, t_k) \\ 0 & \text{for } \ P \neq (x_i, y_j, t_k). \end{cases}$$

The basic functions L_{ijk} are linear in x, y, t.

Hence on $q_{ij}(k)$, $k = 0, 1, \ldots, K-1$:

$$\frac{\partial}{\partial t} U'(x, y, t) = \frac{1}{h^2}\,[(x_{i+1} - x)(y_{j+1} - y)(U_{\bar{t}})_{i+1,j+1}(k) +$$
$$+ (x_{i+1} - x)(y - y_j)(U_{\bar{t}})_{i+1,j}(k) + (x - x_i)(y_{j+1} - y)(U_{\bar{t}})_{i,j+1}(k) +$$
$$+ (x - x_i)(y - y_j)(U_{\bar{t}})_{ij}(k)\,].$$

So we have:

$$\int_{\Omega} \frac{\partial}{\partial t}(U_h'(x, y, t))\,\psi(x, y)\,dx\,dy = \int_{\Omega_h^*\setminus\Omega_h} \frac{\partial}{\partial t}(U_h'(x, y, t))\,\psi(x, y)\,dx\,dy +$$

$$+ h^2 \sum_{\Omega_h} (\Delta_h\,\varphi(U_{ij}(k)) + a_{ij}(k)) \int_0^1\!\!\int_0^1 d_{ij}(r, s)\,dr\,ds + \tag{4.5}$$

$$+ h^2 \sum_{\Gamma_h} (U_{\bar{t}})_{ij}(k) \int_0^1\!\!\int_0^1 d_{ij}(r, s)\,dr\,ds.$$

Here for $(x_i, y_j, t_k) \in \Omega_h$

$$d_{ij}(r, s) = (1 - r)(1 - s)\psi(x_i + rh,\ y_j + sh) +$$
$$+ r(1 - s)\,\psi(x_{i-1} + rh,\ y_j + sh) + rs\,\psi(x_{i-1} + rh,\ y_{j-1} + sh) +$$
$$+ (1 - r)s\,\psi(x_i + rh,\ y_{j-1} + sh).$$

If $(x_i, y_j, t_k) \in \Gamma_h$ some of the terms of the right-hand side are zero. Returning to (4.5) and taking into account the Lipschitz continuity of u_1 and the way U was extended beyond Ω_h, we get:

$$\left| \int_{\Omega_h^*\setminus\Omega} \frac{\partial}{\partial t}(U_h'(x, y, t)\,\psi(x, y)\,dx\,dy \right| < C_1 h \max_{\bar{\Omega}} |\psi| \tag{4.6}$$

$$\left| h \sum_{\Gamma_h} (U_{\bar{t}})_{ij}(k)\; h \int_0^1\!\!\int_0^1 d_{ij}(r, s)\,dr\,ds \right| \le Ch \max_{\bar{\Omega}} |\psi|. \tag{4.7}$$

Now,

$$h^2 \sum \varphi_{x\bar{x}}(U_{ij}(k))\,d_{ij} = h \sum_{j=1}^{N-1} h \sum_{i=m(j)+1}^{M(j)-1} \varphi_{x\bar{x}}(U_{ij}(k))\,d_{ij} =$$
$$= -h \sum_{j=1}^{N-1} h \sum_{i=m(j)}^{N(j)-2} \varphi_x(U_{ij}(k))\,(d_{ij})_x + h \sum_{j=1}^{N-1} \varphi_x(U_{M(j)-1,j}(k))\,d_{M(j)-1,j} -$$
$$- \varphi_x(U_{m(j),j}(k))\,d_{m(j),j}).$$

In view of the definition of d_{ij} and recling that $\psi \in C_0^\infty(\Omega)$:

$$|d_{ij}|,\ |(d_{ij})_x| \le C_2 \max \left\{ |\psi|,\ \left|\frac{\partial\psi}{\partial x}\right|,\ \left|\frac{\partial\psi}{\partial y}\right| \right\} \quad \text{for any } i, j$$

and

$$\max_{\Gamma, j, i} \{|d_{M(j)-1,j}|,\ |d_{m(j),j}|\} < C_3 h \max \left|\frac{\partial\psi}{\partial x}\right|.$$

Using Theorem 4.1

$$h^2 \sum_{j=1}^{N-1} \sum_{i=m(j)}^{M(j)-2} |\varphi_x(U_{ij}(k))\,(d_{ij})_x| < C_4 \max \left\{ |\psi|,\ \left|\frac{\partial\psi}{\partial x}\right|,\ \left|\frac{\partial\psi}{\partial x}\right| \right\}. \tag{4.8}$$

Similar considerations take place for

$$h^2 \sum_{\Omega_h} \varphi_{y\bar{y}}(U_{ij}(k))\,d_{ij}.$$

So if we take into account the continuous imbedding $H^3(\Omega) \subset C^1(\bar{\Omega})$, estimates (4.6) — (4.8) prove our theorem.

5. Convergence of the discrete solution

This section contains the main result of the paper formulated in Theorem 5.2. We also give here some functional analytic results which we need in our proofs.

Consider three Banach spaces B_0, B_1, B_2, satisfying the following algebraic and topological inclusions:

$$B_0 \subset B \subset B_1,$$

B_0, B_1 reflexive.

Let

$$W = \left\{ v;\ v \in L^p(0, T; B_0),\ \ \frac{dv}{dt} \in L^q(0, T; B_1) \right\},$$

$0 < T < +\infty$, p, $q > 1$. If we endow W with the norm

$$\|v\|_{L^p(0,T;B_0)} + \left\| \frac{dv}{dt} \right\|_{L^q(0,T;B_1)},$$

W becomes a Banach space (obviously $W \subset L^p(0, T; B)$).

LEMMA 5.1. *Suppose the imbedding $B_0 \subset B_1$ is compact and that $1 < p$, $q < \infty$. Then $W \subset L^q(0, T; B)$ is also compact.*

For the *proof* see Lions [8].

LEMMA 5.2. *Suppose* $\psi \in \mathfrak{D}(\Omega \times [0, T[)$. *Then for* h, τ *sufficiently small*

$$(5.1)\qquad \int_Q (\tilde{U}\tilde{\psi}_t - \tilde{\varphi}_x(U)\,\tilde{\psi}_x - \tilde{\varphi}_y(U)\tilde{\psi}_y + \tilde{a}\tilde{\psi})\,dx\,dy\,dt + $$

$$+ \int_\Omega \tilde{u}_0(x, y)\,\tilde{\psi}(x, y, 0)\,dx\,dy = 0.$$

Here U is a solution of (2.1) — (2.3).

Proof: Recall that, e.g., $\tilde{\psi}_t$ is the (0)-extension of $(\psi_t)_{ij}(k)$. Multiplying each equation of (2.1) by the corresponding values of ψ we get after summation:

$$\tau h^2 \sum_{k=1}^{k} \sum_{\Omega_h} [U_{\bar{t}}(k) - \Delta_h\,\varphi(U(k)) - a(k)]\,\psi(k) = 0.$$

Hence if τ is taken so small that $\psi(K) = 0$,

$$(5.2)\qquad \tau h^2 \sum_{k=0}^{k-1} \sum_{\Omega_h^+} [U_h(k)\,\psi_t(k) - \varphi_x(U(k))\,\psi_x(k) - \varphi_y(U(k))\,\psi_y(k) -$$

$$- a(k)\,\psi(k)] + h^2 \sum_{\Omega_h} u_0\,\psi(0) = 0.$$

Now if h, τ are so amall that

$$\operatorname{supp}\psi,\ \operatorname{supp}\psi_x,\ \operatorname{supp}\psi_y \subset \Omega_h \times [0, T[,$$

(5.2) becomes identical to (5.1).

LEMMA 5.3. (Lions [8]) *Let D be a bounded domain in* $\mathbf{R}^n$ *and* $u_j, u \in L^p(D)$, $p > 1$, $j = 1, 2, \ldots$. *Suppose that:*

(*i*) $\qquad ||u_j||_p \leq C \quad j = 1, 2, \ldots$

with the constant C independent of j.

(*ii*) $\qquad u_j \to u$ *a.e. on D.*

Then

$$u \to u \text{ weakly in } L^p(D).$$

LEMMA 5.4. *Suppose that* $D \subset \mathbf{R}^n$ *is a bounded domain and that the sequence* $\{u_j\} \subset C(\bar{D})$, $j = 1, 2, \ldots$ *has the following properties:*

(*i*) $\max_{D} |u_j| \leq C \quad j = 1, 2, \ldots$

(*ii*) $u_j \to u$, *a.e. on D.*

(*iii*) *There exists* $q > 1$ *such that* $u \in L^q(D)$. *Then the sequence contains a subsequence convergent in* $L^p(D)$ *for any* $p \in [1, +\infty[$ *and the limit* $u \in L^\infty(D)$.

Proof: The previous lemma ensures that $u_j \to u$ weakly in $L^q(D)$. Because of (*i*) there is a subsequence $\{u_i\} \subset \{u_j\}$ such that

$$u_i \to u \text{ weakly in } L^p(D),$$

for any finite $p \geq 1$ and $u \in \bigcap_{p \geq 1} L^p(D)$. Since there exists a constant C_1 independent of p such that $||u||_p < C_1$ it follows that $u \in L^\infty(D)$.

On the othe hand, according to Egorov's theorem there exists a small measurables set $D_0 \subset D$ such that $u_i \to u$ uniformly on $D \setminus D_0$. Thus for $p \geq 1$:

$$\int_D |u_i - u|^p\,dx = \int_{D/D_0} |u_i - u|^p\,dx + \int_{D_0} |u_i - u|^p\,dx.$$

Since

$$\left(\int_{D_0} |u_i - u|^p\,dx\right)^{1/p} \leq \left(\int_{D_0} |u_i|\,dx\right)^{1/p} + \left(\int_{D_0} |u|^p\,dx\right)^{1/p}$$

the weak convergence of the sequence and the absolute continuity of the integral implies that $u_i \to u$ in $L^p(D)$. In what follows a bar over the subscripts denotes a suitably chosen subsequence.

THEOREM 5.1. *Suppose that*

(*i*) *Condition* (*A*) *holds*

(*ii*) u_1 *is Lipschitz continuous in all variables*

(*iii*) *U is the solution of problem* (2.1) — (2.3)

(*iv*) (u_1, Ω) *has the property* (P_h) *or* (P).

Then there exists a subsequence $\{\bar{h}, \bar{\tau}\} < \{h, \tau\}$ *and a function* $v \in L^\infty(Q)$, $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \in L^2(Q)$ *such that:*

(*j*) $(\varphi_x(U_{\bar{h}}))' \to v$ *in* $L^q(Q)$ *for* $q \in [1, \infty[$

(*jj*) $(\varphi_x(U_{\bar{h}}))' \to \frac{\partial v}{\partial x}$, $(\varphi_y(U_{\bar{h}}))' \to \frac{\partial v}{\partial y}$ *weakly in* $L^2(Q)$

(*jjj*) $\varphi(\tilde{U}_h) \to v$ *a.e. and* $v \geq 0$ *a.e. on Q.*

Proof: Recall first that U has been defined on Ω_h^* and the extension U' also on $[0, T]$. From Theorem 4.1 and (*ii*) we have then, that:

$$(5.3)\qquad \tau h^2 \sum (\varphi_x(u)^2 + \varphi_y(u)^2) < C.$$

Consequently taking into account Theorem 2.1, for a constant C_1 we get:

$$(5.4)\qquad \int_{Q_h^+} ((\varphi(U)')^2 + (\varphi_x(U)')^2 + (\varphi_y(U)^2)')\,dx\,dy\,dt < C_1$$

and also

$$(5.5)\qquad \int_{Q_h^+} ((\varphi(U)')^2 + (\varphi_x(U)_{(1)})^2 + (\varphi_y(U)_{(2)})^2)dx\,dy\,dt < C_1.$$

Here Q_h^+ is the rectangular domain generated by the mesh-points of the set denoted in the same way.
From (5.5) we get for a subsequence of indeces $\{\bar{h}, \bar{\tau}\}$:

$$\varphi(U_{\bar{h}})' \to v,\ \varphi_x(U_{\bar{h}})_{(1)} \to v_1,\ \varphi_y(U_{\bar{h}})_{(2)} \to v_2,$$

weakly in $L^2(Q)$. Now, according to Theorem 3.1:

$$\varphi_x(U_{\bar{h}})' \to v_1,\ \varphi_y(U_{\bar{h}})' \to v_2, \text{ weakly in } L^2(Q).$$

On the other hand,

$$(5.6)\qquad (\varphi_x(U))_{(1)} = \frac{\partial \varphi(U)'}{\partial x},\qquad (\varphi_y(U))_{(2)} = \frac{\partial \varphi(U)'}{\partial y},$$

which implies

$$v_1 = \frac{\partial v}{\partial x},\qquad v_2 = \frac{\partial v}{\partial y}$$

and also (*jj*). This completes the proof.

Next, let us take in Lemma 5.1, $p = q = 2$, $B_0 = H^1(\Omega)$, $B = L^r(\Omega)$, $r \geq 1$ and finite, $B_1 = H^{-3}(\Omega)$. Then by Theorem 4.2 and (5.6), $\varphi(U_{\bar{h}})'$ is precompact in $L^2(0, T; L^r(\Omega))$. Then a subsequence (denoted in the same way) of $\varphi(U_{\bar{h}})'$, converges to $v \in L^2(Q)$ in $L^2(Q)$. This enables us to aply Lemma 5.4 which proves (*j*). This entails, passing if necessary to another subsequence, (*jjj*). The proof is completed.

Next, we introduce a new continuous extension of the discrete function U defined on the mesh points of $\bar{Q}_h$. Namely

$$U_h^- = \varphi^{-1}(\varphi(U_h)').$$

This is a continuous interpolate of U over the rectangular domain $\bar{Q}_h$ which belongs to $H^1(Q_h)$.

THEOREM 5.2. *Under the hypotheses of Theorem 5,1, there exists a function $u \in L^\infty(Q)$, $u \geq 0$ a.e., with*

$$\frac{\partial \varphi(U)}{\partial x},\ \frac{\partial \varphi(U)}{\partial y} \in L^2(Q)$$

such that for a suitable subsequence of indices $\{\bar{h}, \bar{\tau}\}$:

(i) $\varphi(U_{\bar{h}})' \to \varphi(u)$ *in* $L^p(Q)$ *for any* $p \in [1, \infty[$

(ii) $\varphi_x(U_{\bar{h}})' \to \dfrac{\partial \varphi(u)}{\partial x}$, $\varphi_y(U_{\bar{h}})' \to \dfrac{\partial \varphi(u)}{\partial y}$ *weakly in* $L^2(Q)$

(iii) $U_{\bar{h}}' \to u$ *weakly in* $L^p(Q)$, $1 < p < \infty$ *and a.e.* $h, \tau \to 0$.

Moreover if there exists a constant $x_0 \geq 0$ such that $\varphi(x) \geq x$ for $x \geq x_0$, then

(v) $U_{\bar{h}}^- \to u$ *in* $L^p(Q)$ *for any* $1 \leq p < +\infty$.

Proof: Suppose v is the function of Theorem 5.1 and $u = \varphi^{-1}(v)$. Clearly, (i), (*ii*) hold. Since $v \in L^\infty(Q)$, $u \in L^\infty(Q)$ and by Lemma 5.3 $U_{\bar{h}}' \to u$ weakly in $L^p(Q)$, which proves (*iii*). The solution U of (2.1) — (2.3) satisfies (5.1) for $\psi \in \mathfrak{D}(\Omega \times [0, T[)$ and h, τ sufficiently small. Since

$$\tilde{\psi}_t \to \frac{\partial \psi}{\partial t},\ \tilde{\psi}_x \to \frac{\partial \psi}{\partial x},\ \tilde{\psi}_y \to \frac{\partial \psi}{\partial y},\ \tilde{\psi} \to \psi$$

$\tilde{\psi}(x, y, 0) \to \psi(x, y, 0)$, uniformly in strictly interior domains, we see from (5.1) if we take into account (*i*), (*ii*), that u is a solution of (1.4). Finally if $\varphi(x) \geq x$ for $x \geq x_0$, then $\varphi^{-1}(x) \leq x$, so that

$$\varphi^{-1}(x) \leq x + x_0, \text{ for } x \geq 0.$$

This implies that

$$\varphi^{-1}(\varphi(U_{\bar{h}})') \to u \text{ in } L^p(Q) \text{ for any } p \in [1, +\infty[.$$

6. Initial and boundary conditions. Uniquness of aproximating sequence.

LEMMA 6.1. (Lions [8]) *If X is a Banach space and $f \in L^p(0, T; X)$, $df/dt \in L^p(0, T; X)$, $1 \leq p < +\infty$, then f (after eventually changing it on a set of measure zero of $]0, T[$) is a continuous mapping.*

CONSEQUENCE 6.1. *Suppose u is the solution constructed in Theorem 5.1. Then $u(0) = u|_{t=0}$ makes sense.*

Indeed if we take in (1.4)

$$f(x, y, t) = f_1(x, y)f_2(t)$$

with $f_1 \in \mathfrak{D}(\Omega)$, $f_2 \in \mathfrak{D}(0, T)$ it becomes:

$$\left(\left(u, \frac{df_2}{dt}\right), f_1\right) = -\left(\left(\frac{\partial \varphi(u)}{\partial x}, f_2\right), \frac{\partial f_1}{\partial x}\right) - \left(\left(\frac{\partial \varphi(u)}{\partial y}, f_2\right), \frac{\partial f_1}{\partial y}\right) +$$

$$+ ((a, f_2), f_1) = ((\Delta\varphi(u), f_2), f_1) + ((a, f_2), f_1).$$

This implies that equation (1.1) is also fulfilled in the sense of distributions. Because

$$\frac{\partial \varphi(u)}{\partial x},\ \frac{\partial \varphi(u)}{\partial y} \in L^2(Q),$$

we have

$$\frac{du}{dt} = \Delta\varphi(u) + a \in L^2(0,\ T;\ H^{-1}Q)).$$

and Lemma 6.1 can be applied with $X = (H^{-1}(\Omega)$.

The following lemma is a particular case of Theorem 2.1 from Lions [7, Vol. II].

LEMMA 6.2. *Let $f \in L^2(0,\ T;\ H^1(\Omega))$ and suppose Ω to be sufficiently regular. Then the trace $f|_S$ exists and belongs to $H^{1/2,0}(S)$.*

Moreover the mappiug: $u \to u|_S$, $H^{1,0}(Q) \to H^{1/2,0}(S)$ is continuous.

Here

$$H^{1/2,0}(S) = L^2(0,\ T);\ H^{1/2}(\partial\Omega)) \subset L^2(S).$$

THEOREM 6.1. *If Ω is sufficiently regular and conditions of Theorem 5.1 hold, than u possesses a trace $u|_S$.*

Proof: Since $\varphi(u) \in H^{1,0}(Q)$, by Lemma 6.2.

$$\varphi(u)|_S \in H^{1/2,0}(S) \subset L^2(S).$$

On the other hand $\varphi^{-1}(x) \leq x + x_0$ so that $u|_S$ also exists and the trace operator is continuous.

THEOREM 6.2. *Suppose that Ω is sufficiently regular. Assume that u is the solution constructed in Theorem 5.2 and that the conditions of this theorem are fulfilled. Then,*

$$u|_S = u_1$$

and $U_{\bar{h}}|_S \to u|_S$, uniformly on S.

Proof: According to Theorem 5.2 there exists a sequence $\{U_n\} \subset \subset \{U_{\bar{h}}\}$, $n = 1, 2, \ldots, U_n \subset C(\bar{Q})$ such that

$$U_n \to u \text{ in } L^p(Q),\ p \in [1, \infty[.$$

At the same time there is another sequence $V_n \in C^\infty(Q)$ such that

$$\text{(6.1)} \qquad \max_{\bar{Q}} |U_n - V_n| < \frac{1}{n}, \qquad n = 1, 2, \ldots.$$

Since according to Theorem 6.1 the trace operator is continuous:

$$V_n|_S \to u|_S \text{ in } L^2(S)$$

which implies in view of (6.1):

$$U_n|_S \to u|_S \text{ in } L^2(S).$$

On the other hand u_1 being Lipschitz continuous and by the way u_2 was constructed, $U_n|_S \to u|_S$ uniformly.

THEOREM 6.3. *Assume that conditions of Theorem 5.2 are fulfilled. Then $u|_{t=0}$ exists and*

(*i*) $u|_{t=0} = u_0 \quad x \in \Omega$

(*ii*) $U_{\bar{h}}(x, y, 0) \to u_0(x, y)$ *uniformly on* $\bar{\Omega}$.

Proof: By Consequence 6.1, $u|_{t=0}$ exists and belongs to $L^2(\Omega)$. Using the same sequence as in the proof of the previous theorem we get:

$$U_n(x, y, 0) \to u_0(x, y) \text{ on } \bar{\Omega}.$$

The proof is complete.

A better understanding of how (1.3) is fulfilled and a more precise characterization of the regularity of Ω is given in Theorem 6.4. But first we formulate:

LEMMA 6.3. *Suppose $\Omega \subset \mathbf{R}^2$ is a bounded domain such that its frontier can be divided in a finite number of arcs whose tangents make with either Ox or Oy an angle greater than a positive constant.*

Then for any function $V: \bar{Q}_h \to \mathbf{R}$ and any sufficiently small $r < 0$, there exist constants A and B, independent of h, τ, such that:

$$\text{(6.2)} \qquad \tau h^2 \sum_{S_{r,h}} V^2(k) \leq A r^2 \tau h^2 \sum_{S_{r,h}} (V_x^2(k) + V_y^2(k)) + Br\, h\tau \sum V(k).$$

Here

$$S_{rh} = Q_h^+ \cap S_r,\quad S_r = \{M \in \bar{Q};\, d(M, S) \leq r\},\ S_h = \Gamma_h \times \{1, 2, \ldots, K\}.$$

The *proof* is given in [2].

THEOREM 6.4. *Suppose Ω satisfies conditions of Lemma 6.3. Assume that u is the solution obtained in Theorem 6.2. Let $f \in C(\bar{Q})$ be a Lipschitz continuous function such that:*

$$f \geq 0 \text{ and } f|_S = u_1.$$

Then

$$\text{(6.3)} \qquad \frac{1}{r}\int_{S_r} (\varphi(u) - \varphi(f))^2\, dx\, dy\, dt \to 0 \text{ as } r \to 0.$$

Proof: As in the proof of Theorem 6.2 we use the sequence $\{U_n\}_{n=1}^{\infty}$. Recall that $U_n \in C(\bar{Q})$ and $U_n \to u$ in $L^2(Q)$, $n \leq \infty$.

Let $\{f_n\}$ be the corresponding sequence of discretizations of f.

Admit that (6.3) is not true. Then there exists a constant $C > 0$ such that

$$\text{(6.4)} \qquad \frac{1}{r}\int_{S_r} (\varphi(u) - \varphi(f))^2\, dx\, dy\, dt > C,$$

for a sequence of numbers r, converging to 0.

According to (6.2):

$$\tau h^2 \sum_{S_{rh}} (\varphi(U_n) - \varphi(f_n))^2 \leq A r^2 \tau h^2 \sum_{S_{rh}} [(\varphi(U_n) - \varphi(f_n))_x^2 +$$

$$+ \varphi(U_n) - \varphi(f_n))_y^2] + Brh \sum_{S_h} (\varphi(U_n) - \varphi(f_n))^2.$$

But $|u_2 - f_n| < Lr$, so that

$$\frac{h^2\tau}{r}\sum_{S_{rh}} (\varphi(U_n) - \varphi(f_n))^2 \leq Dr,$$

D independent or h, τ, r (and n).

Consequently for $\rho < r$,

$$\frac{1}{r}\int_{S_r \setminus S_\rho} (\varphi(U_u)' - \varphi(f_n)')\, dx\, dy\, dt \leq K_1 r$$

K_1 having the same properties as K.
Now letting $\rho \to 0$, we get

$$\frac{1}{r}\int_{S_r} (\varphi(U_n)' - \varphi(f_n)')^2\, dx\, dy\, dt \leq K_1 r.$$

Finally for h, $\tau \to 0$ ($n \to \infty$),

$$\frac{1}{r}\int_{S_r} (u - f)^2\, dx\, dy\, dt \leq K_1 r,$$

which contradicts (6.3).

Remark 6.1. The existence of the function f is ensured by condition (P). This condition can be weakened by supposing that u_1 is Lipschitz continuous as it was proved in [1].

THEOREM 6.5. *The whole sequence U_h^τ tends for h, $\tau \to 0$ to the unique solution u, provided that conditions of Theorem 6.2 are fulfilled.*

This is immediately seen from the uniqeuness of the limit of the sequence, u.

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