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AN ALGORITHM BASED ON THE GRAPH THEORY FOR SOLVING THE TIME-CRITERION ASSIGNEMENT PROBLEM

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1. In [2] the author presented an algorithm for solving the following time-criterion assignement problem *TCAP*: let $E_1, E_2, \ldots, E_n$ be n sending centers, each of them having one unit from a product and let $D_1, D_2, \ldots, D_n$ be n receiving centers, each asking one unit from the same product (the product unit being indivisible).

Let T be the square matrix

(1) $$T = (t_{ij})_{i,j=\overline{1,n}} \qquad t_{ij} \geq 0$$

where t_{ij} represents the necessary time for shipping of the product unit from E_i to D_j. Find the matrix

(2) $$X = (x_{ij})_{i,j=\overline{1,n}} \qquad \text{where}$$

$$x_{ij} = \begin{cases} 1 & \text{if it is a transportation between } E_i \text{ and } D_j \\ 0 & \text{contrarily} \end{cases}$$

so that the total time allocated to the whole program be minimum.

Same as in the price-criterion assignement problem, each admissible solution of *TCAP* is a Boolean matrix with exactly n free elements equal to 1 (that means each element belong to different rows and columns).

The problem is to find that admissible solution $\bar{X}$ of a *TCAP* for which

(3) $$t_{\bar{X}} = \min_{X \in \mathfrak{X}} \{\max\,(t_{1j_1}, t_{2j_2}, \ldots, t_{nj_n})\}$$

where $(j_1, j_2, \ldots, j_n)$ is a permutation of $\{1, 2, \ldots, n\}$, t_{ij} being the corresponding times of those components of the solution X which are equal to 1.

In the present note, the author gives a new algorithm for solving *TCAP* using the graph theory.

2. Let be the *TCAP* problem. We can attach to this problem the graph (GA) by using the general means of the associated graph to a transportation problem [1]. Then:

PROPOSITION 1. *The sum of the kernels of a graph associated to a transportation problem, for two valuations X and Y, X and Y being two different simple solutions of the transportation problem, is a graph which contains at least a circle.*

PROPOSITION 2. *The necessary and sufficient condition for a graph to be even, is that its set of vertexes could be grouped in two clases, so that every edge of the graph links vertexes from different classes.*

The definitions and theorems from [1] are true in the assignement problem, too. But from the characteristics of the problem, one can prove other properties, as follows:

LEMMA 1. *All admissible solution of assignement problem are simple; if the graph (GA) has 2n vertexes, the admissible solutions number is n!*

LEMMA 2. *The sum of the kernels of the graph (GA) for the valuations X and Y, where X and Y are two different solutions of the assignement problems, is a graph which contains at least one even circle.*

LEMMA 3. *Let X be an admissible solution of an assignement problem and N(X) the kernel of the graph (GA) corresponding to the solution X. If at N(X) one attaches an even circle, so that every edge of the circle links an E with a D vertex, in each vertex of the circle one meets exactly two edges, then by remove of the edges of N(X), implicated in the respective circle, one obtains the kernel of a new admissible solution Y associated with X.*

Using these results one can prove:

THEOREM 1. *Starting at an admissible solution X of the assignation problem, one can arrive (using the procedure described in the Lemma* 3) *to any other admissible solution Y of the considerated problem.*

Proof. From the Lemma 2, if X and Y are two different admissible solutions of the assignement problem, $N(X) + N(Y)$ contains at least one even circle.

If it contains exactly one, by the procedure, which was presented in the Lemma 3, we can arrive at Y from X in a single step.

If it contains k circles, with the same procedure we can build up a string of admissible solutions of *TCAP*, $X_1, X_2, \ldots, X_{k+1}$, where

$$X_1 = X, \quad X_{k+1} = Y$$

and $N(X_i) + N(X_{i+1})$ contains exactly one even circle.

3. Now we can give an algorithm for solving of *TCAP*. We shall work with the time matrix (1) of the considerated problem, the direct treatment on the graph being more difficult.

Let X be an admissible solution of *TCAP*. We superpose the matrix X over the matrix T, enclosing the elements t_{ij}, corresponding to $x_{ij} = 1$, and let unenclosed the other elements, t_{ij}. Thus for each admissible solution, on each row and column, it will exist exactly one enclosed element t_{ij}, corresponding to the edge (E_i, D_j), from the kernel of the solution X. These t_{ij} lies on different rows and columns, and so they are free. We shall note this matrix by T_X.

Starting from the matrix T_X, in a circle built as in the Lemma 3 and replacing the unenclosed vertexes, one obtains a new matrix Y, with n free enclosed elements, corresponding to a new admissible solution Y.

Using this procedure we can pass from an admissible solution to another, and the maximum times

$$(4) \qquad t_X = \max_{(i,j)\in\{(i,j)/x_{ij}=1\}} t_{ij}$$

should be decreased and in this way after a finite number of steps we arrive to the solution having minimum t_X, which is optimum solution of *TCAP*.

Using this remark, we enunciate the following algorithm for optimizing *TCAP*:

1. *One determines an admissible solution X of the problem* (for example by Hungarian method);

2. *One computes* t_X *which correspond to* X, *by* (4);

3. *One rewrites the matrix T, enclosing each elements* t_{ij}, *corresponding to* $x_{ij} = 1$, *from* X, *and letting empties* (we mark with a dot) *any other positions* (i, j), *where* $t_{ij} \geqslant t_X$, *so it is obtained the matrix* T_X.

4. *It is marked* (by an asterisk) *the enclosed elements* t_{kh} *for which* $t_{kh} = t_X$.

If there exist more than one element of this kind t_{kh} *we mark only one of them. If, starting from the marked* t_{kh}, *we can build up a circle (CX), so than none from its vertex coincides with an empty position from* T_X*; than enclosing the unenclosed vertexes of the circle and viceversa, the matrix* X *has the elements* $x_{ij} = 1$, *corresponding to the enclosed elements* t_{ij} *and the other elements equal with zero, which is a new admissible solution of TCAP. The algorithm restarts after this at the step 2. If we can't build up this circle, the solution is optimum and the algorithm ends.*

For this algorithm we can prove the following theorem:

Theorem 2. The above described algorithm is finite and the obtained solution is optimum.

Proof. There are a finite number of admissible solutions of *TCAP* (from the Lemma 1, n!). So, from the 4-th step of the algorithm, there exists a finite number of possibilities to build a new admissible solution of $TCAP$ associated with the starting solution Also, because for each pair of succesive solutions X_1 and X_2 we have

$$t_{X_1} > t_{X_2} \tag{5}$$

(if $t_{X_1} = t_{X_2}$, at each iteration is eliminated a vertex (p, q) for which $t_{pq} = t_{X_1}$, there is in a finite number of steps one arrive at (5)), in this algorithm none of solution can't repeat and after a finite number of steps we arrive at the situation that the algorithm can't returned, thus the algorithm is finite.

We assume now that the obtained solution isn't optimum. Result that there exist a solution Y, so that $t_Y < t_X$ and Y is different by X at least two components different at 0.

If we asume that Y is different by X by only one, result that if i_1 is the row index of these components $x_{i_1j_1}$ and $y_{i_1k_1}$, then X has at the column k_1 a component different by zero $x_{i_2k_1}$, where $i_2 \neq i_1$ (which result from the $TCAP$ admissible solutions structure). This is a component of Y too, because X and Y are different by only one element of this kind. From this Y has on the column k_1 two elements different by zero, which contradicts its admissibility.

There result that Y is different by X, with r components differents by zero, where $r > 1$.

Building the circle which links tese r vertexes of X with the r components of Y, so that each edge links an occupated vertex from X with one from Y, we obtaine an even circle. By $t_Y < t_X$ result that X isn't optimum.

4. For exemplification of the algorithm we recall an example from [2], which is the $TCAP$, defined by the time matrix:

$$T = \begin{pmatrix} 8 & 5 & 7 & \boxed{2} & 10 \\ 13 & 9 & \boxed{6} & 4 & 8 \\ 1 & 5 & 4 & 8 & \boxed{3} \\ \boxed{2} & 11 & 7 & 4 & 5 \\ 7 & \boxed{2} & 3 & 9 & 8 \end{pmatrix}$$

1. By Hungarian method we obtain

$$X_1 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and $t_X = \max\{2, 6, 3, 2, 2\} = 6$.

3. One build

$$T_{X_1} = \begin{pmatrix} \cdot & 5 & \cdot & \boxed{2} & \cdot \\ \cdot & \cdot & \boxed{6}^{*} & 4 & \cdot \\ 1 & 5 & 4 & \cdot & \boxed{3} \\ \boxed{2} & \cdot & \cdot & 4 & 5 \\ \cdot & \boxed{2} & 3 & \cdot & \cdot \end{pmatrix}$$

4. One marks in T_{X_1} the element $t_{23} = 6$. Starting from this we build (CX_1), on T_{X_1}. Enclosing the unenclosed vertexes of the circle and with the reverse we obtain

$$X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and the algorithm is restarted at the step 2.

2. $t_{X_2} = \max\{5, 4, 3, 2, 2\} = 5$

3.

$$\begin{pmatrix} \cdot & \boxed{5}^{*} & \cdot & 2 & \cdot \\ \cdot & \cdot & \cdot & \boxed{4} & \cdot \\ 1 & 5 & 4 & \cdot & \boxed{3} \\ \boxed{2} & \cdot & \cdot & 4 & 5 \\ \cdot & 2 & \boxed{3} & \cdot & \cdot \end{pmatrix}$$

One mark in T_2 the element $t_{12} = 5$.

4. One remark that we can't build up another circle, corresponding to t_{12}, X_2 which is an optimum solution of the problem, where $t_{X_1} = 5$ is the minimum time in which we can perform the whole transportation using this solution.

Remark. Building the circle (CX_2) on T_{X_2}, we can obtain a new optimum solution :

$$X_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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