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A FAMILY OF SIXTH-ORDER INTERATIVE METHODS FOR SOLVING EQUATIONS

by

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In this paper an one — parameter family of sixth — order methods for the iterative solution of nonlinear operator equations of the form:

(1) $$P(x) = 0$$

where P maps the Banach space X into a Banach space Y, is described. For non—zero values of the parameter, each application of each family member requires no explicit evaluations of derivatives. The convergence of these methods is proved under several assumptions, and a numerical example is given.

1. An one — parameter family of sixth-order iterative methods

In order to obtain solutions of (1), we consider a family of iterative methods defined by the formula:

(2) $$x_{n+1} = y_{n+1} - [P(y_{n+1}, z_{n+1})]^{-1} P(y_{n+1})$$

where:

$$z_{n+1} = y_{n+1} - [3I - \Gamma_n(P(x_n, y_{n+1}) + P(y_{n+1}, v_n))]\Gamma_n P(y_{n+1});$$

$$y_{n+1} = x_n - \Gamma_n P(x_n); \quad \Gamma_n = [P(x_n, v_n)]^{-1};$$

$$v_n = x_n - cP(x_n);$$

$P(x, y)$ — first divided difference of $P(x)$ [1], c — real parameter, I — unit operator.

Conditions of the convergence of processes (2) are given by the following.

THEOREM. *Let us assume that the following conditions are fulfilled :*

1) *Equation* (1) *possesses a solution* x^* *which belongs to sphere :*

$$S_0 = \{x : ||x - x_0|| \leqslant d\}, \quad d > 0;$$

2)
$$||[P(x', x'')]^{-1}|| \leqslant B; \quad ||P(x', x'')|| \leqslant N;$$
$$||P(x', x'', x''')_1|| \leqslant M; \quad ||P(x', x'', x''')_2|| \leqslant M;$$
$$||P(x', x'', x''')_1 - P(x'''', x'', x')_1|| \leqslant L_1||x' - x''''|| + L_2||x''' - x'||$$

for every x', x'', x''', x'''' *of sphere* $S = \{x : ||x - x_0|| \leqslant (1 + \alpha)d\}$, *where :*

$$\alpha = \max\{1 + |c|N, \ BM(1 + |c|N)d, \ KBM(1 + |c|N)d^3\};$$

$P(x, y, z)_1$, $P(x, y, z)_2$ — *second-order divided differences of* $P(x)$ [1];

3) $md < 1$, *where :*

$$m = \sqrt[5]{B^3M^3K(1 + |c|N)^2};$$

$$K = B^2M^2N(2B + |c|)(1 + BN + |c|N) + B^2M^2(1 + |c|N) + BN(B + |c|)(L_1BN + L_2(1 + BN)).$$

Then the equation (1) *has a unique solution* x^* *in* S_0, *the sequence* $\{x_n\}$ *defined by* (2) *converges to* x^* *and the rate of convergence is given by the following inequality :*

(3)
$$||x_n - x^*|| \leqslant \frac{1}{m}(md)^{6^n}$$

Proof. The theorem is proved by mathematical induction. It is easy to see that $||x_0 - x^*|| \leqslant d$, $x_0 \in S$ and we observe that the inequality (3) holds for $n = 0$. We new obtain by induction that

$$||x_n - x^*|| \leqslant \frac{1}{m}(md)^{6^n} \text{ and } ||x_n - x_0|| \leqslant (1 + m^5d^5)d.$$

In fact, if these are true up to some $n \geqslant 1$, then using the Newton interpolating formula [1], the conditions of theorem and (2) we get :

$$x_{n+1} - x^* = [P(y_{n+1}, z_{n+1})]^{-1}P(y_{n+1}, z_{n+1}, x^*)_1(z_{n+1} - x^*)(y_{n+1} - x^*);$$

$$z_{n+1} - x^* = \{\Gamma_nP(x_n, y_{n+1}, v_n)_2\Gamma_nP(x_n, x^*)(x_n - x^*)\Gamma_n[P(x_n, v_n) - P(y_{n+1}, x^*)] + \Gamma_nP(y_{n+1}, v_n, x^*)_1\Gamma_n[P(x_n, v_n) - P(x_n, x^*)](x_n - x^*) + \Gamma_n[P(y_{n+1}, v_n, x^*)_1 - P(x_n, v_n, y_{n+1})_1][\Gamma_n - cI]P(x_n, x^*)(x_n - x^*) + \Gamma_nP(x_n, v_n, y_{n+1})_1[\Gamma_n - cI]P(x_n, x^*)(x_n - x^*)\Gamma_n[P(x_n, v_n) - P(y_{n+1}, x^*)]\}(y_{n+1} - x^*);$$

$$y_{n+1} - x^* = \Gamma_nP(x_n, v_n, x^*)_1[I - cP(x_n, x^*)](x_n - x^*)(x_n - x^*);$$

from which it follows that :

(4)
$$||x_{n+1} - x^*|| \leqslant B^3M^3K(1 + |c|N)^2||x_n - x^*||^6 = m^5||x_n - x^*||^6$$

and

$$||x_{n+1} - x_0|| \leqslant ||x_{n+1} - x^*|| + ||x^* - x_0|| \leqslant (1 + m^5d^5)d.$$

Thus the sequence $\{x_n\}$ defined by (2) remains in S, and $\lim_{n\to\infty} x_n = x^*$

From (4) we obtain (3).

Finally, we prove that the equation (1) has a unique solution x^* in S_0. Suppose that x^{**} is a solution of equation (1) in S_0 and $x^{**} \neq x^*$. The same proof shows that $\lim_{n\to\infty} x_n = x^{**}$. By the uniqueness of the limit point of convergent sequence $\{x_n\}$, it follows that $x^* \equiv x^{**}$ and hence the uniqueness of x^* in S_0 could have been concluded from that fact.

Thus the theorem is completely proved.

REMARK 1. One special case of (2) is that for $c = 0$. We have :

(5)
$$x_{n+1} = y_{n+1} - [P(y_{n+1}, z_{n+1})]^{-1}P(y_{n+1})$$

where :

$$z_{n+1} = y_{n+1} - [3I - \Gamma_n(P(x_n, y_{n+1}) + P(y_{n+1}, x_n))]\Gamma_nP(y_{n+1});$$
$$y_{n+1} = x_n - \Gamma_nP(x_n); \quad \Gamma_n = [P'(x_n)]^{-1}$$

REMARK 2. The iterative process proves useful for finding solutions of $f(x) = 0$, where f is a real-valued function of one variable. In this case, we obtain, from (2) and (5), the following methods :

(6)
$$x_{n+1} = y_{n+1} - \frac{z_{n+1} - y_{n+1}}{f(z_{n+1}) - f(y_{n+1})} f(y_{n+1})$$

where :

$$z_{n+1} = y_{n+1} - \frac{f(x_n)f(v_n) + f(v_n)f(y_{n+1}) + f(x_n)f(y_{n+1})}{f(v_n)[f(x_n) - f(v_n)]} cf(y_{n+1});$$

$$y_{n+1} = x_n - \frac{v_n - x_n}{f(v_n) - f(x_n)} f(x_n); \quad v_n = x_n - cf(x_n); \quad c \neq 0;$$

and

(7)
$$x_{n+1} = y_{n+1} - \frac{z_{n+1} - y_{n+1}}{f(z_{n+1}) - f(y_{n+1})} f(y_{n+1})$$

where :

$$z_{n+1} = y_{n+1} - \left(1 + 2\frac{f(y_{n+1})}{f(x_n)}\right)\frac{f(y_{n+1})}{f'(x_n)};$$

$$y_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

It is of interest to note that (6) is a very economical family of sixth-order methods, each requiring only four function evaluations per iteration, and that (7) requires no more than three function and one derivative evaluations per iteration.

2. Numerical illustration.

We now give a numerical example, using the formula (6), which not only illustrates the method practically but also serves to check the validity of the theoretical results we have derived.

Let us find the solution of the equation:

$$f(x) \equiv x^3 - 2x - 5 = 0$$

by the Newton–Raphson method and the method (6) with $c = 0.1$, using the starting value $x_0 = 2$. The root correct to 13 decimal places is [2]:

$$x^* = 2.0945514815423.$$

The successive approximations are set down in table 1.

Table 1

The Newton–Raphson method	The method (6) with $c = 0.1$
$x_1 = 2.1$	$x_1 = 2.0945514$
$x_2 = 2.0945681$	
$x_3 = 2.0945514$	

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