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INTRODUCING OF THE NOTION OF DEVELOPMENT WITH RESPECT TO A SCALE IN A COMMUTATIVE RING WITH UNITY

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In this paper we consider a commutative ring with unity and, by means of two properly chosen subrings, we will introduce the notions of negligible element related to another one, scale, development of an element with respect to a given scale. Then, we give calculation rules for the introduced developments. Finally, we illustrate, by a convenient particularization of the mentioned theory, that we can find again the asymptotical developments of the functions.

Let A be a commutative ring, with unity element, where we note the nul element by „0", and the unity element „1". Let N and C be two subrings of A, so as C should not contain divisors of zero and so as the following conditions should be satisfied:

(1) $$1 \in C$$

(2) $$N \cap C = \{0\}$$

(3) $$C \cdot N \subset N$$

Remark. From (1) and (2) it follows

(4) $$1 \notin N.$$

DEFINITION 1. *We say that element* $x \in A$ *is N-negligible related to* $y \in A$, *if exists* $\nu \in N$, *so as* $x = \nu y$. *We shall note this by* $x = n(y)$.

PROPOSITION 1. *For any* $x \in A$, $x \neq 0$ *and x different from a divisor of zero, takes place* $x \neq n(x)$.

If exists $\nu \in N$, so as $x = \nu x$, then, due to the evident relation $x = 1x$, would result $\nu x = 1x$, that is $(\nu - 1)x = 0$. As x is not a divisor of zero, and $x \neq 0$, it results that $\nu = 1$, this contradicting the relation (4).

PROPOSITION 2. *If $x, y \in A$ and one of them is different from 0 and different from a divisor of zero, then at the most one of the relations: $x = n(y)$, $y = n(x)$ might take place.*

Should $x \neq 0$ and x different from a divisor of zero, in order to fix the ideas. If the two relations of enunciation take place simultaneously, then we shall have $\nu_1, \nu_2 \in N$, so as $x = \nu_1 y$ and $y = \nu_2 x$. From these relations it results $x = \nu_1(\nu_2 x) = (\nu_1\nu_2)x$, that's to say $x = n(x)$, that would contradict proposition 1.

PROPOSITION 3. *If $x = n(y)$ and $y = n(z)$, then $x = n(z)$.*

From hypotheses it results that exists $\nu_1, \nu_2 \in N$, so as $x = \nu_1 y$ and $y = \nu_2 z$. From these relations we deduce $x = (\nu_1\nu_2)z$ and, as $\nu_1\nu_2 \in N$, it results that $x = n(z)$.

PROPOSITION 4. *If $x_1 = n(y)$ and $x_2 = n(y)$, then $x_1 + x_2 = n(y)$.*

From the hypothesis it results that $\nu_1, \nu_2 \in N$ exists, so as $x_1 = \nu_1 y$ and $x_2 = \nu_2 y$. We deduce $x_1 + x_2 = (\nu_1 + \nu_2)y$ and, as $\nu_1 + \nu_2 \in N$, the conclusion results.

PROPOSITION 5. *If $x = n(y)$ and $c \in C$, then $cx = n(y)$.*

From the hypothensis it results that $\nu \in N$ exists so as $x = \nu y$. From this, $cx = c(\nu y) = (c\nu)y$ and as according to relation (3) $c\nu \in N$, the conclusion results.

PROPOSITION 6. *If $x_1 = n(y_1)$ and $x_2 = n(y_2)$, then $x_1x_2 = n(y_1y_2)$.*

According to the hypothesis, $\nu_1, \nu_2 \in N$ exists, so as $x_1 = \nu_1 y$ and $x_2 = \nu_2 y$. From these two relations, it results that $x_1x_2 = \nu_1\nu_2 y_1 y_2$ and, as $\nu_1\nu_2 \in N$, the conclusion results.

DEFINITION 2. *We name N-scale, a subset $E \subset A$, which forms a semigroups whith respect to the multiplication from A, which does not contain the nul element or divisors of zero from A and which has the following property: if $e', e'' \in E$, $e' \neq e''$ then at least one of the conditions $e' = n(e'')$ or $e'' = n(e')$ should take place.*

THEOREM 1. *A N-scale is a semigroup ordered by the relation*

$$(5) \qquad x\alpha y \quad \text{when} \quad x = n(y).$$

Demonstration. Should E be a N-scale. From definition 2 and proposition 2 it results that for any $e', e'' \in E$, $e' \neq e''$, exists one and only one of the relations $e' = n(e'')$ or $e'' = n(e')$. Taking into considerations propositions 1 and 3, it results that the relation „α" is one of a strict ordering in E. In order to show the compatibility of the relation introduced, by the multiplicating operation from E, we consider e, e', e'' of E, $e'\alpha e''$. It results that $\nu \in N$ exists, so as $e' = \nu e''$. Multiplying both membres of this equality by e, we find $ee' = \nu e e''$, $ee' = n(ee'')$, that is to say $ee'\alpha ee''$.

PROPOSITION 7. *The relation $ce = n(e)$, where $c \in C$, $e \in E$ implies $c = 0$.*

According to the hypothesis, $\nu \in N$ exists, so as $ce = \nu e$, that is to say $(c - \nu)e = 0$. As $0 \notin E$ and E does not contain divisors of zero, it results that $c - \nu = 0$, that is to say $c = \nu$. According to relation (2), it results $c = 0$.

DEFINITION 3. *We name principal part of the element $x \in A$ with respect to N-scale E, the element ce, where $c \in C$, $c \neq 0$ $e \in E$, which satisfies the relation*

$$(6) \qquad x = ce + n(e).$$

THEOREM 2. *If the element $x \in A$ has a principal part with respect to N-scale E, then this principal part is unique.*

Demonstration. We suppose that exists $c' \in C$, $c' \neq 0$ and $e' \in E$ so as

$$(7) \qquad x = c'e' + n(e').$$

The relations (6) and (7) imply

$$(8) \qquad ce + n(e) = c'e' + n(e').$$

Supposing that $e' \neq e$ and, for fixing the ideas, should be $e' = n(e)$. On the basis of propositions 5 and 3 it results that both $c'e'$ and $n(e')$ are N-negligible related to e. Taking into consideration proposition 4, relation (8) implies $ce = n(e)$, which, according to proposition 7, is possible only when $c = 0$, which contradicts the hypothesis. In conclusion, the supposition $e' = n(e)$ leads to a contradiction. In a similar way we emphasize that, relation $e = n(e')$ can not take place. So to say, $e' \neq e$ can not take place. Replacing e' by e, within relation (8), we find $(c - c')\,e = n(e)$, which, on the basis of proposition 7, implies $c = c'$. By this we have demonstrated the uniqueness of the principal part ce.

DEFINITION 4. *We name development of the element $x \in A$ with respect to the N-scale E, of e_s precizion, a reprezentation of x under the form*

$$(9) \qquad x = \sum_{k=1}^{s} c_k e_k + n(e_s)$$

where for any natural numbers i, j, $1 \leqslant i < j \leqslant s$, $e_i, e_j \in E$ and $e_j \alpha e_i$ and where for $k = 1, 2, \ldots, s$, $c_k \in C$, $c_k \neq 0$.

THEOREM 3. *The development of an element $x \in A$, with respect to the N-scale E, of e_s precision, is unique.*

Demonstration. From the development (9) it results that c_1e_1 is the principal part of x, c_2e_2 is the principal part of $x - c_1e_1$ and generally, c_ke_k $k = 3, 4, \ldots, s$, is the principal part of $x - c_1e_1 - c_2e_2 - \ldots - c_{k-1}e_{k-1}$ with respect to the N-scale E. On the basis of this remark and on the basis of the uniqueness of the principal part, garanted by theorem 2, results the uniqueness of the development of a given precision, with respect to the N-scale E.

The procedure used in the demonstration of theorem 3 may serve for the effective finding of a development with respect to a N-scale.

Let be $y \in A$ and its development with respect to the N-scale E, of the precision e'_t

$$y = \sum_{l=1}^{t} d_l e'_l + n(e'_t) \tag{10}$$

Knowing the developments (9) and (10) of the x respectively y, with respect to the N-scale E, we may obtain the development of the sum $x + y$ and of the product xy, with respect to the same N-scale.

In order to obtain the development of the $x + y$ sum, should add up, member by member, the (9) and (10) equalities, then they should arrange the terms of the obtained sum in a decreasing order of the elements from E, operation possible due to theorem 1. As the sum should contain the terms $n(e_s)$ and $n(e'_t)$, the precision of the devlopment should be max (e_s, e'_t).

In order to obtain the development of the xy product, we should multiply, member by member, the (9) and (10) equalities. The products $e_i e'_j$, $i = 1, 2, \ldots, s$, $j = 1, 2, \ldots, t$ so obtained should belong to E. According to theorem 1, these products may be arranged in a decreasing order. The sum of the terms under the form $e'_j n(e_s)$, $j = 1, 2, \ldots, t$ and $e_i n(e'_t)$, $i = 1, 2, \ldots, s$ will be negligible related to max $(e_1 e'_t, e_s e'_1)$ and so, the precision of the development of this product will by equal to this maximum.

We shall particularize the theory exposed so as to find again the asymptotical developments of the functions. In order to do this, we shall consider the following sets:

A = the set of the functions defined over a neighbourhood of point $\alpha \in \mathbf{R}$;

C = the set of constant functions;

N = the set of the functions which in point α have the limit 0;

Evidently, A is a ring, C and N are subrings of A, C does not contain divisors of zero and the conditions (1), (2) and (3) are satisfied.

The relation $f = n(g)$, introduced by the definition 1, is reduced in such a case, to the relation $f = o(g)$, where „o" is the notation of Landau.

Should E be an asymptotical scale, in the clasical sense of the word, known in analysis, that is to say a subset of A, having the properties

(i) All the functions of E are positive

(ii) In point α, every function from E has the limit 0 or ∞.

(iii) The product of two functions from E belongs to E. If $f \in E$, and if λ is a real number, then $f^\lambda \in E$ (in particular, it results that the quotient of two functions of E, belongs to E).

They can easy check that a development of an element from A, with respect to the N-scale E, development introduced by definition 4, reduces itself to an usual asymptotical development of a function, with respect to the asymptotical scale E.

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