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THE ANNULUS OF STARLIKENESS FOR UNIVALENT FUNCTIONS

by

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1. Let S be the class of functions $f(z) = z + a_2z^2 + \ldots$, $f(0) = 0$, $f'(0) = 1$ wich are regular and univalent in the unit disk, $|z| < 1$.

It is known [2] that any circle $|z| = r$, $r \leqslant r^* = \operatorname{th}\frac{\pi}{4} = 0{,}65\ldots$ is mapped by any function of the class S into a starlike curve C_r with respect to the origin. If $r > r^*$ there are functions in the class S for which C_r is not stalike, but certain arcs of the curve C_r are starlike with respect to the origin.

The aim of this paper is to find, for a given r with $r > r^*$, the extremal annulus (wich we shall call *annulus of starlikeness*) such that for any function in the class S the arcs of C_r which lie outside this annulus are starlike with respect to the origin. Since S is a compact class, there exists this extremal annulus. We will determine this extremal annulus by using the variational method of SCHIFFER – GOLUZIN [1].

2. Analytically, our problem can be written as follows: for a given r, $r > r^*$ find $\max\limits_{\varphi\in S} |\varphi(z)|$, respectively $\min\limits_{\varphi\in S} |\varphi(z)|$, where $|z| = r$ and

$$\operatorname{Re}\frac{z\varphi'(z)}{\varphi(z)} = 0.$$

Let $|z| = r$ and let $f \in S$ with $\operatorname{Re}\frac{zf'(z)}{f(z)} = 0$ be the extremal function, for which $\max\limits_{\varphi\in S} |\varphi(z)|$ is attained.

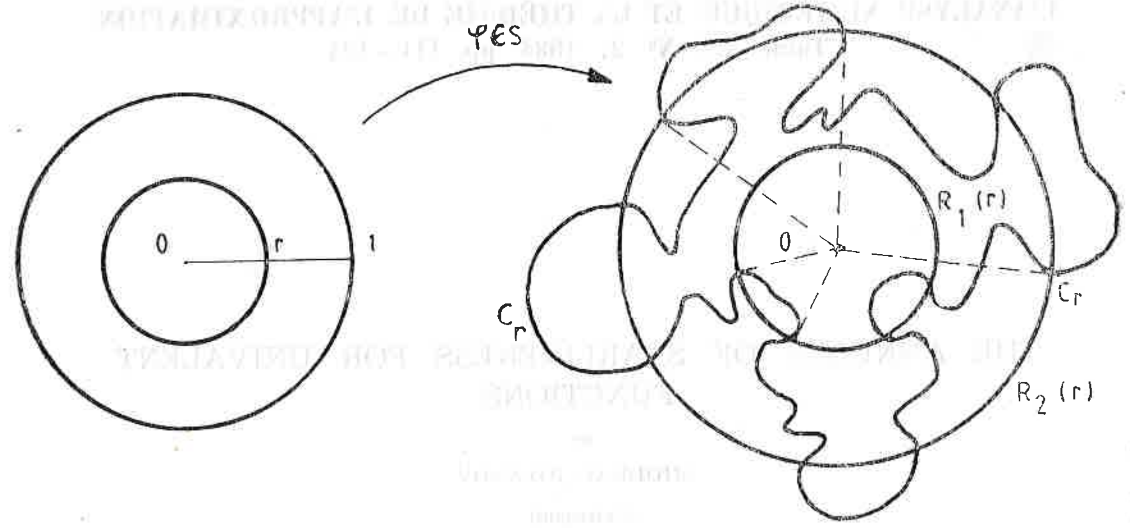

Fig. 1. Geometrical illustration of the annulus of starlikennes.

Now we consider a variation of the function f given by the Schiffer—Goluzin's formula [1]:

(1) $$f^*(z) = f(z) + \lambda V(z;\ \zeta;\ \psi) + 0(\lambda^2), \quad |\zeta| < 1, \ \lambda > 0,$$

ψ real number, where

$$V(z;\ \zeta;\ \psi) = e^{i\psi}\frac{f^2(z)}{f(z) - f(\zeta)} - e^{i\psi} f(z)\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2 -$$
$$- e^{i\psi}\frac{zf'(z)}{z-\zeta}\cdot\zeta\cdot\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]^2 + e^{-i\psi}\frac{z^2 f'(z)}{1-\bar{\zeta} z}\,\bar{\zeta}\cdot\overline{\left[\frac{f(\zeta)}{\zeta f'(\zeta)}\right]}^2.$$

It is known [1] that for a sufficiently small λ, the function $f^*(z)$ belongs to the class S.

Next we consider a variation z^* of z:

$$z^* = z + \lambda h + 0(\lambda^2), \quad h = \left.\frac{\partial z^*}{\partial \lambda}\right|_{\lambda=0}$$

which satisfies the conditions

(2) $$|z^*| = r \text{ and } \mathrm{Re}\,\frac{z^* f^{*\prime}(z^*)}{f^*(z^*)} = 0.$$

We have

$$|z^*|^2 = |z|^2 + 2\lambda\ \mathrm{Re}\,(\bar{z}h) + o(\lambda^2) = r^2.$$

Since $|z| = r$, from the above relation we obtain

(3) $$\mathrm{Re}\,(\bar{z}h) = 0.$$

We can consider z a real and positive number, because the condition $\mathrm{Re}\,\frac{zf'(z)}{f(z)} = 0$ is invariant under rotations.

Therefore our initial condition becomes $\mathrm{Re}\,\frac{f'(r)}{f(r)} = 0$ and the relation (2) can be written in the following form:

$$\mathrm{Re}\,h = 0 \quad \text{or} \quad h + \bar{h} = 0.$$

Replacing z by z^* in the expresion of $f^*(z)$ we have

$$\frac{z^* f^{*\prime}(z^*)}{f^*(z^*)} = A + \lambda B + O(\lambda^2)$$

where $A = \frac{zf'(z)}{f(z)}$, $B = \frac{hf'(z) + zhf''(z) + zV_z'(z;\ \zeta;\ \psi)}{f(z)} - \frac{zf'(z)}{f^2(z)}[hf'(z) + V(z;\ \zeta;\ \psi)]$.

Using the relation (2), we obtain the condition

$$\mathrm{Re}\,B = 0 \quad \text{or} \quad B + \bar{B} = 0.$$

Since $\bar{h} = -h$, from the above relation we obtain

(4) $$h = \frac{\bar{D} + D}{\bar{C} + C}$$

where

$$C = \frac{f'(z) + zf'(z)}{f(z)} - \frac{zf'(z)}{f^2(z)}$$

and

$$D = \frac{zV_z'(z;\ \zeta;\ \psi)}{f(z)} - \frac{zf'(z)\ V(z;\ \zeta;\ \psi)}{f^2(z)}.$$

Since $f(z)$ is extremal, we have

$$|f^*(z^*)| \leqslant |f(z)|,$$

which is equivalent to

$$|f(z)|^2 + 2\lambda\mathrm{Re}\{\overline{f(z)}\,[hf'(z) + V(z;\ \zeta;\ \psi)]\} + O(\lambda^2) \leqslant |f(z)|^2$$

which yields

(5) $$\mathrm{Re}\,\{\overline{f(z)}\,[hf'(z) + V(z;\ \zeta;\ \psi)]\} \leqslant 0.$$

By using (4) and the following notations

$$z = r,\ f = f(z),\ w = f(\zeta),\ l = f(r),\ m = f''(r)$$

$$V = V(r;\ \zeta;\ \psi), \qquad V' = V'_r(r;\ \zeta;\ \psi),$$

the condition (5) becomes

$$\text{Re}\left[2\left(\frac{rV'}{f} - \frac{rlV}{f^2}\right)n + \frac{V}{f}\right] \leqslant 0, \tag{6}$$

where

$$n = -\frac{1}{2 + r\left(\frac{m}{l} + \frac{\overline{m}}{\overline{l}}\right)}.$$

We note that n is real and that $2 + r\left(\frac{m}{l} + \frac{\overline{m}}{\overline{l}}\right) = 2\left[1 + \text{Re}\,\frac{rf''(r)}{f'(r)}\right]$.

We first, suppose that the point at which the tangent from the origin is drawn is not an inflexion point, that is

$$1 + \text{Re}\,\frac{rf''(r)}{f'(r)} \neq 0 \text{ or, equivalently } 2 + r\left(\frac{m}{l} + \frac{\overline{m}}{\overline{l}}\right) \neq 0.$$

Relation (6) can also be written

$$\text{Re}\,(xV + yV') \leqslant 0, \tag{7}$$

where

$$x = \frac{f - 2rln}{f^2} \text{ and } y = \frac{2rn}{f}.$$

Replacing the expressions of V and V' in (7) we obtain

$$\text{Re}\,[e^{i\psi}(E - GF)] \leqslant 0, \tag{8}$$

where

$$E = \frac{f^2 - (f + 2rnl)w}{(f - w)^2},$$

$$G = 1 + \frac{xrl\zeta}{r-\zeta} + \frac{y[r(r - \zeta) - l\zeta]}{(r-\zeta)}\,\zeta - \frac{\bar{x}r^2\bar{l}\,\zeta}{1 - r\zeta} - \frac{\bar{y}[r^2\overline{m}(1 - r\zeta) + r\,\bar{l}\,(2 - r\zeta)]\zeta}{(1 - r\zeta)^2}$$

and

$$F = \left(\frac{w}{\zeta w'}\right)^2.$$

Since ψ is arbitrary, from (8) we deduce $E - GF = 0$, By replacing E, G and F it follows that the extremal function $w = f(\zeta)$ must satisfy the following equation:

$$\left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{f^2 - (f + 2 \cdot r \cdot n \cdot l)w}{(f - w)^2} = \frac{\sum_{k=0}^{4} t_k \zeta^k}{(r - \zeta)^2(1 - r\zeta)^2}, \tag{9}$$

where

$t_0 = r^2,$

$t_1 = -2r - 2r^3 + xlr^2 + ymr^2 - \bar{y}\overline{m}r^4 - \bar{x}\bar{l}r^4 - 2\bar{y}\bar{l}r^3,$

$t_2 = r^4 + 4r^2 + 1 - r(2r^2 + 1)(ym + xl) + r^3(r^2 + 2)(\bar{y}\overline{m} + \bar{x}\bar{l}) + r^2(r^2 + 4)\bar{y}\bar{l} - yl,$

$t_3 = -2r - 2r^3 - r^2(2r^2 + 1)(\bar{y}\overline{m} + \bar{x}\bar{l}) + r^2(r^2 + 2)(ym + xl) - 2r(1 + r^2)\bar{y}\bar{l} + 2 \cdot ryl,$

$t_4 = r^2[1 + r(\bar{y}\overline{m} + \bar{x}\bar{l} - ym - xl) + \bar{y}\bar{l} - yl].$

Using the expression of n and the fact that $\text{Re}\,\frac{l}{f} = 0$ it is easily shown that $t_4 = r^2$, $t_1 = t_3$ and t_2 is real number.

3. It may be shown [1] that the extremal function $w = f(\zeta)$ maps the unit disk onto the entire w — plane slit along a finite number of analytic arcs. Let q be the point which is mapped into an end—point of a slit. The polinominal $t_0 + t_1\zeta + t_2\zeta^2 + t_3\zeta^3 + t_4\zeta^4$ has the double root. Since $t_0 = t_4 = r^2$ and $\zeta = q$ is the double root for the above polynominal it follows that the equation (9) may be written

$$\left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{f^2 - (f + 2 \cdot rnl)w}{(f - w)^2} = \frac{r^2(1 - \bar{q}\zeta)^2(1 - 2 \cdot kq\zeta + q^2\zeta^2)}{(r - \zeta)^2\,(1 - r\zeta)^2} \tag{10}$$

where k is real number.

Let $q = e^{i\theta}$, where θ is a real number.

Making $\zeta \to r$ in (10) and taking the real and imaginary parts we obtain the relations:

$$\frac{nl}{f} = \frac{i(k - 1)\,\sin\theta}{1 - r^2} \tag{11}$$

and

$$k = \frac{2r[2r\cos\theta - (1 + r^2)]\cos\theta + (1 - r^2)^2}{2r[-2r + (1 + r)^2\cos\theta]} \tag{12}$$

respectively.

Let $P(\zeta)$ be the right-side member of the differential equation (10) i.e.

$$P(\zeta) = \frac{r^2(1-\bar{q}\zeta)^2(1-2kq\zeta+q^2\zeta^2)}{(r-\zeta)^2(1-r\zeta)^2}.$$

On $|\zeta| = 1$ we have $P(\zeta) \geqslant 0$ ([3]). Let $\zeta = e^{i\gamma}$, where γ is a real number. The expression of $P(\zeta)$ may be written in the following form:

$$P(\zeta) = \frac{4[\cos(\gamma-\theta)-1]\cdot[\cos(\gamma+\theta)-k]}{\left[2\cos\gamma - \left(r+\frac{1}{r}\right)\right]^2}.$$

Since $P(\zeta) \geqslant 0$, for any ζ, $|\zeta| = 1$ from (11) we obtain $k > 1$.

Since $k > 1$, we deduce that θ satisfies the condition

$$\cos\theta > \frac{2r}{1+r^2}. \tag{13}$$

We remark that $1-2kq\zeta+q^2\zeta^2 = 0$ has the roots:

$\zeta_{1,2} = (k \pm \sqrt{k^2-1})\bar{q}$. Letting $\rho = k - \sqrt{k^2-1}$, we obtain $\zeta_1 = \rho\bar{q}$ and $\zeta_2 = \frac{1}{\rho}\cdot\bar{q}$. We note that $|\zeta_1| < 1$ and $|\zeta_2| > 1$.

Taking $\zeta = \zeta_1$ in the equation (10) we obtain

$$w(\zeta_1) = \frac{f^2}{f+2rnl}. \tag{14}$$

4. The equation (10) can be written in the following form

$$\left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{f^2-(f+2\cdot rnl)w}{(w-f)^2} = \frac{(\zeta-q)^2(\zeta-\rho\bar{q})\left(\zeta-\frac{1}{\rho}\bar{q}\right)}{(\zeta-r)^2\left(\zeta-\frac{1}{r}\right)^2}. \tag{15}$$

By integrating the above equation we obtain that the extremal function $w = f(\zeta)$ is given implicitly by the equation:

$$\begin{cases} \dfrac{\rho\bar{q}}{1-\rho^2}\,\dfrac{f+2\cdot rnl}{f^2}\left(\dfrac{\sqrt{-2\cdot rnlf}+f}{\sqrt{-2rnlf}-f}\right)^{\sqrt{-2\cdot r\frac{nl}{f}}}\cdot\dfrac{f+\sqrt{f^2-(f+2\cdot rnl)w}}{f-\sqrt{f^2-(f-2rnl)w}}\times \\ \times\left(\dfrac{\sqrt{-2\cdot rnlf}-\sqrt{f^2-(f+2\cdot rnl)w}}{\sqrt{-2\cdot rnlf}+\sqrt{f^2-(f+2\cdot rnl)w}}\right)^{\sqrt{-2r\frac{nl}{f}}} = \dfrac{1+\rho}{1-\rho}\left(\dfrac{\sigma+\rho}{\sigma-\rho}\right)^{s_1}\times \\ \times\left(\dfrac{\tau+\rho}{\tau-\rho}\right)^{s_2}\cdot\dfrac{1-s}{1+s}\cdot\dfrac{\rho+s}{\rho-s}\cdot\left(\dfrac{\sigma-s}{\sigma+s}\right)^{s_1}\cdot\left(\dfrac{\tau-s}{\tau+s}\right)^{s_2}, \end{cases} \tag{16}$$

where:

$$\sigma = \sqrt{\frac{r\rho-\bar{q}\rho^2}{r\rho-\bar{q}}},\quad s_1 = \frac{(r-\bar{q}\rho)(r-q)}{\sigma(1-r^2)},$$

$$\tau = \sqrt{\frac{\rho-r\bar{q}\rho^2}{\rho-r\bar{q}}},\quad s_2 = \frac{(r\rho-q)(r-\bar{q})}{\tau(1-r^2)},\quad s = \sqrt{\frac{\zeta-\rho\bar{q}}{\zeta-\frac{1}{\rho}\bar{q}}}.$$

Taking $\zeta = \zeta_1 = \rho\bar{q}$ in (16) and using the fact that $f^2 - f(+2\cdot rnl)w(\zeta_1) = 0$, we obtain

$$f(r) = \frac{\rho\bar{q}}{(1+\rho)^2}\cdot\left(1+2r\frac{nl}{f}\right)\cdot\left(\frac{\sqrt{-2r\frac{nl}{f}}+1}{\sqrt{-2r\frac{nl}{f}}-1}\right)^{\sqrt{-2r\frac{nl}{f}}}\cdot\left(\frac{\sigma-\rho}{\sigma-\rho}\right)^{s_1}\cdot\left(\frac{\tau-\rho}{\tau+\rho}\right)^{is_2}. \tag{17}$$

Taking $\zeta \to r$ in (15) we obtain

$$s_1 = \sqrt{-2r\frac{nl}{f}},\quad s_2 = -is_1 \tag{18}$$

and the expression of $f(r)$ becomes

$$f(r) = \frac{\rho\bar{q}}{(1-\rho)^2}\cdot(1-s_1^2)\cdot\left(\frac{s_1+1}{s_1-1}\right)^{s_1}\cdot\left(\frac{\sigma-\rho}{\sigma+\rho}\right)^{s_1}\cdot\left(\frac{\tau+\rho}{\tau-\rho}\right)^{is_1}. \tag{19}$$

Since ρ, q, s_1, σ and τ in the relation (19) depend on θ (r is given) it follows that $f = f(r)$ given by (19) depends on θ.

5. For determining $|f(r)|$ as a function of θ we notice that

$$|f(r)| = \frac{\rho}{(1+\rho)^2}\cdot|u_1|\cdot|u_2|\cdot|u_3|\cdot|u_4| \tag{20}$$

where

$$u_1 = 1-s_1^2,\quad u_2 = \left(\frac{s_1+1}{s_1-1}\right)^{s_1},\quad u_3 = \left(\frac{\sigma-\rho}{\sigma+\rho}\right)^{s_1},\quad u_4 = \left(\frac{\tau+\rho}{\tau-\rho}\right)^{is_1}. \tag{21}$$

We shall distinguish two cases:

a) if $\theta \in [0, \pi]$, we obtain

$$|f(r)| = \frac{\rho}{(1+\rho)^2}\sqrt{1+4a^4}\cdot\left[\frac{1+4a^4}{(2a^2-2a+1)^2}(x_1^2+y_1^2)(x_2^2+y_2^2)\right]^{\frac{1}{2}}\cdot e^{iU_1} \tag{22}$$

where:

$$a = \sqrt{\frac{(k-1)\sin\theta}{1+r^2}}$$

(23)

$$x_1 = \frac{\dfrac{N_1}{N_2} - \rho}{\dfrac{N_1}{N_2} + \sqrt{2}\,N_1 \cdot \sqrt{\dfrac{\rho}{N_1 N_2}} + A + \rho}, \quad y_1 = \frac{\sqrt{2}\,N_1 \sqrt{\dfrac{\rho}{N_1 N_2}} - A}{\dfrac{N_1}{N_2} + \sqrt{2}\,N_1 \sqrt{\dfrac{\rho}{N_1 N_2}} + A + \rho}$$

$$x_2 = \frac{\dfrac{N_2}{N_1} - \rho}{\dfrac{N_2}{N_1} - \sqrt{2}\,N_2 \sqrt{\dfrac{\rho}{N_1 N_2}} + A + \rho}, \quad y_2 = \frac{-\sqrt{2}\,N_2 \sqrt{\dfrac{\rho}{N_1 N_2}} - A}{\dfrac{N_2}{N_1} - \sqrt{2}\,N_2 \sqrt{\dfrac{\rho}{N_1 N_2}} + A + \rho}$$

$$U_1 = \operatorname{arctg} \frac{(2a^2-1)(x_2 y_1 - x_1 y_2) + 2a(x_1 y_2 + y_1 y_2)}{(2a^2-1)(x_1 x_2 + y_1 y_2) + 2a(x_1 y_2 + y_1 x_2)}$$

$$N_1 = \sqrt{\frac{\rho}{r^2\rho^2 - 2r\rho\cos\theta + 1}}, \quad N_2 = \sqrt{\frac{\rho}{r^2 - 2r\rho\cos\theta + \rho^2}}$$

$$A = \rho(1+r^2) - r(1+\rho^2)\cos\theta.$$

b) if $\theta \in (\pi, 2\pi)$, we obtain

(24) $$|f(r)| = \frac{\rho}{(1+\rho)^2}^{\sqrt{1+4b^4}} \left[\frac{1+4b^4}{(2b^2-2b+1)^2} \cdot \frac{x_1^2+y_1^2}{x_2^2+y_2^2}\right]^{\frac{b}{2}} \cdot e^{-bU_2}$$

where

(25) $$\begin{cases} b = \sqrt{\dfrac{(k-1)\cdot(-\sin\theta)}{1-r^2}} \\ U_2 = \operatorname{arctg} \dfrac{(2b^2-1)(x_2 y_1 + x_1 y_2) + 2b(-x_1 x_2 + y_1 y_2)}{(2b^2-1)(x_1 x_2 - y_1 y_2) + 2b(x_1 y_2 + x_2 y_1)} \end{cases}$$

and the expressions of x_1, y_1, x_2 and y_2 are the same as in the case *a)*.

6. For obtaining θ which occurs in the expression of $|f(r)|$ given by (22) and (24) we shall use (16) and the fact that n is a real number.

Taking $\zeta \to r$ in (16) we obtain

(26) $$\begin{cases} n = \dfrac{(\rho - rq)(r\rho - \bar{q})}{2r(1-\rho^2)} \left(1 + 2r\dfrac{nl}{f}\right)\left(\dfrac{\tau+\sigma}{\tau-\sigma}\right)^{-i} \times \\ \times \left[\dfrac{\left(\sqrt{-2r\dfrac{nl}{f}} + 1\right)(\sigma-\rho)(1+\sigma)}{\left(\sqrt{-2r\dfrac{nl}{f}} - 1\right)(\sigma+\rho)(1-\sigma)}\right]^{\frac{1}{s_1}} \end{cases}$$

where $\dfrac{nl}{f}$ is given in (11)

Because n is a real number it results that $\operatorname{Im}(n) = 0$ or $\arg(n) = n_0\pi$ where n_0 is an integer. The equation in θ will be given by

(27) $$\operatorname{Im}\left\{\frac{(\rho - rq)(r\rho - \bar{q})}{2r(1-r^2)} \cdot \left(1 + 2r\frac{nl}{f}\right) \cdot \left(\frac{\tau+\sigma}{\tau-\sigma}\right)^{-i} \times \left[\frac{\left(\sqrt{-2r \cdot \dfrac{nl}{f}} + 1\right) \cdot (\sigma-\rho)(1+\sigma)}{\left(\sqrt{-2r\dfrac{nl}{f}} - 1\right) \cdot (\sigma+\rho)(1-\sigma)}\right]^{\frac{1}{s_1}}\right\} = 0.$$

Since $r > \operatorname{th}\frac{\pi}{4} = 0{,}65\ldots$ and $\cos\theta > \dfrac{2r}{1+r^2}$ it follows that θ must satisfy the condition: $\cos\theta > 0{,}913\ldots$

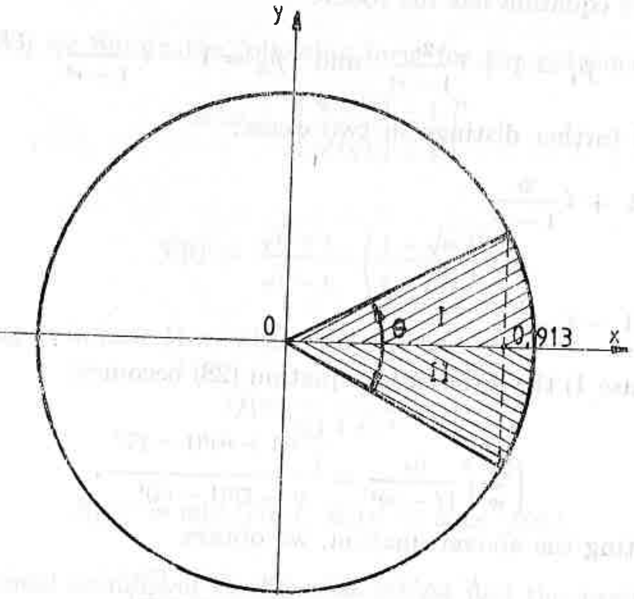

Fig. 2. The possible position of θ.

7. Next we consider the case when the point in which the tangent from the origin is drawn is an inflexion point, that is

$$1 + \mathrm{Re}\,\frac{rf''(r)}{f'(r)} = 0 \quad \text{or} \quad 2 + r\cdot\left(\frac{m}{l} + \frac{\bar{m}}{\bar{l}}\right) = 0.$$

In this case, proceeding like in 2, we obtain the differential equation of the extremal function $w = f(\zeta)$ in the form

$$\left(\frac{\zeta w'}{w}\right)^2 \cdot \frac{lw}{(f-w)^2} = \frac{(d_1 + d_2\zeta + d_3\zeta^2)\cdot\zeta}{(r-\zeta)^2(1-r\zeta)^2} \tag{28}$$

where

$$\begin{cases} d_1 = r(1-r^2)\,\dfrac{l}{f}\cdot p, \quad d_2 = (1-r^4)\,\dfrac{l}{f}\,(1-p), \\ d_3 = r(1-r^2)\,\dfrac{l}{f}\,(p-2), \quad p = r\left(\dfrac{l}{f} - \dfrac{m}{\bar{l}}\right). \end{cases} \tag{29}$$

Since the extremal function has the property enunciated at 3, let $|q| = 1$ such that $w'(q) = 0$. Then the trinomial $d_1 + d_2\zeta + d_3\zeta^2$ has double solution $\zeta = q$. Then condition $d_2^2 - 4\,d_1 d_3 = 0$ becomes:

$$(1-r^2)^2p^2 - 2(1-r^2)^2p + (1+r^2)^2 = 0.$$

The above equation has the roots:

$$p_1 = 1 + i\,\frac{2r}{1-r^2} \quad \text{and} \quad p_2 = 1 - i\,\frac{2r}{1-r^2}.$$

We shall further distinguish two cases:

1) $p = 1 + i\dfrac{2r}{1-r^2}$

and

2) $p = 1 - i\dfrac{2r}{1-r^2}$.

In the case 1) the differential equation (28) becomes:

$$\left(\frac{\zeta w'}{w}\right)^2 \frac{lw}{(f-w)^2} = \frac{\dfrac{l}{f}\,r(1+ir)^2(1-\bar{q}\zeta)^2}{(r-\zeta)^2(1-r\zeta)^2}.$$

Integrating the above equation, we obtain

$$\frac{\sqrt{f}+\sqrt{w}}{\sqrt{f}-\sqrt{w}} = \frac{\sqrt{r}+\sqrt{\zeta}}{\sqrt{r}-\sqrt{\zeta}}\cdot\left(\frac{1+\sqrt{r\zeta}}{1-\sqrt{r\zeta}}\right)^i. \tag{30}$$

From (30) we obtain the extremal function in the form

$$w = f\cdot\left[\frac{\Phi(\sqrt{\zeta})-1}{\Phi(\sqrt{\zeta})+1}\right]^2 \tag{31}$$

where

$$\Phi(t) = \frac{\sqrt{r}+t}{\sqrt{r}-t}\cdot\left(\frac{1+\sqrt{r}\cdot t}{1-\sqrt{r}\cdot t}\right)^i.$$

From (30) we obtain

$$|f(r)| = \frac{r}{1+r^2}. \tag{32}$$

In the case (2) the differential equation (28) becomes

$$\left(\frac{\zeta w'}{w}\right)^2 \frac{lw}{(f-w)^2} = \frac{\dfrac{l}{f}\,r(1-ir)^2(1-\bar{q}\zeta)^2}{(r-\zeta)^2(1-r\zeta)^2}.$$

By integrating the above equation we obtain

$$\frac{\sqrt{f}-\sqrt{w}}{\sqrt{f}+\sqrt{w}} = \frac{\sqrt{r}-\sqrt{\zeta}}{\sqrt{r}+\sqrt{\zeta}}\cdot\left(\frac{1+\sqrt{r\zeta}}{1-\sqrt{r\zeta}}\right)^i. \tag{33}$$

Form (33) we obtain the following form for the extremal function

$$w = f\cdot\left[\frac{\Psi(\sqrt{\zeta})-1}{\Psi(\sqrt{\zeta})+1}\right]^2 \tag{34}$$

where

$$\Psi(t) = \frac{\sqrt{r}+t}{\sqrt{r}-t}\cdot\left(\frac{1-\sqrt{rt}}{1+\sqrt{rt}}\right)^i.$$

Proceding as in case 1) we also obtain in this case

$$|f(r)| = \frac{r}{1+r^2}. \tag{35}$$

8. Let

$$R_1(r) = \min_{f\in S}|f(r)|, \quad R_2(r) = \max_{f\in S}|f(r)|. \tag{36}$$

The extremal annulus of stralikeness will be just the annulus $R_1(r) < |w| < R_2(r)$.

We thus obtain the following result :

THEOREM. *For* $r \in (\text{th}\,\frac{\pi}{4},\ 1)$, $f \in S$, *the values of* $R_1(r)$ *and* $R_2(r)$ *are given by the relation :*

(37) $$R_1(r) = \min\{E_1, E_2, E_3\},\quad R_2(r) = \max\{E_1, E_2, E_3\}.$$

The expressions E_1, E_2, E_3 *have the following form :*

$$E_1 = \frac{\rho}{(1+\rho)^2}\sqrt{1+4a^4}\left[\frac{1+4a^4}{(2a^2-2a+1)^2}(x_1^2+y_1^2)(x_2^2+y_2^2)\right]^{\frac{a}{2}}\cdot e^{aU_1}$$

where

$$U_1 = \operatorname{arctg}\frac{(2a^2-1)(x_2y_1 - x_1y_2) + 2a(x_1x_2+y_1y_2)}{(2a^2-1)(x_1x_2+y_1y_2) + 2a(x_1y_2+y_1x_2)}$$

when θ *occurs in region I in fig. 2,*

$$E_2 = \frac{\rho}{(1+\rho)^2}\sqrt{1+4b^4}\left[\frac{1+4b^4}{(2b^2-2b+1)^2}\cdot\frac{x_1^2+y_1^2}{x_2^2+y_2^2}\right]^{\frac{b}{2}}\cdot e^{-bU_2}$$

where

$$U_2 = \operatorname{arctg}\frac{(2b^2-1)(x_2y_1 + x_1y_2) + 2b(-x_1x_2+y_1y_2)}{(2b^2-1)(x_1x_2+y_1y_2) + 2b(x_1y_2+x_2y_1)}$$

when θ *occurs in region II in fig. 2, and respectively,*

$$E_3 = \frac{r}{1+r^2}.$$

The value of θ *in the expressions of* E_1 *and* E_2 *is given by the equations*

(38) $$\sin(\theta_1 + \theta_2 + \theta_3 + \theta_4) = 0$$

when θ *belongs to the first region in fig 2 and respectively :*

(39) $$\sin(\theta_1 + \theta_2 + \theta_3 + \theta_4') = 0,$$

when θ *belongs to the second region in fig. 2, where :*

$$\theta_1 = \operatorname{arctg}\frac{-2r + (1+r)^2\cos\theta}{(1-r^2)\sin\theta},$$

$$\theta_2 = \operatorname{arctg}\frac{(-2r\cos\theta + 1 + r^2)^2\sin\theta}{(1-r^2)\,[-2r+(1+r^2)\cos\theta]},$$

$$\theta_3 = -\ln\frac{|(1+r^2)k - 2r\cos\theta + \sqrt{4r^2k^2 - 4r(1+r^2)\cos\theta\cdot k + (1-r^2)^2 + 4r^2\cos^2\theta}\,|}{(1-r^2)\sqrt{k^2-1}},$$

$$\theta_4 = \frac{1}{4a}\left[\ln(u^2+v^2) + 2\operatorname{arctg}\frac{v}{u}\right],$$

$$u = u_1u_2u_3 - u_1v_1v_3 - u_2v_1v_3 - u_3v_2v_1,$$

$$v = -v_1v_2v_3 + v_1u_2u_3 + v_2u_3u_1 - v_3u_1u_2,$$

$$u_1 = \frac{2a^2-1}{2a^2-2a+1},\quad v_1 = \frac{2a}{2a^2-2a+1},$$

$$u_2 = \frac{\frac{N_1}{N_2} - \rho}{\frac{N_1}{N_2} + \sqrt{2}\cdot N_1\sqrt{\frac{\rho}{N_1N_2} + A} + \rho},$$

$$v_2 = \frac{\sqrt{2}N_1\sqrt{\frac{\rho}{N_1N_2} - A}}{\frac{N_1}{N_2} + \sqrt{2}N_1\sqrt{\frac{\rho}{N_1N_2} + A} + \rho},$$

$$u_3 = \frac{1 - \rho\frac{N_1}{N_2}}{1 + \rho\frac{N_1}{N_2} - \sqrt{2}N_1\sqrt{\frac{\rho}{N_1N_2} + A}},$$

$$v_3 = \frac{\sqrt{2}N_1\sqrt{\frac{\rho}{N_1N_2} - A}}{1 + \rho\frac{N_1}{N_2} - \sqrt{2}N_1\sqrt{\frac{\rho}{N_1N_2} + A}} \quad\text{and}$$

$$\theta_4' = \frac{1}{4b}\left[-\ln(u^2+v^2) + 2\operatorname{arctg}\frac{v}{u}\right].$$

The variables which appear in the theorem maintain their previous significations.

9. In order to find the extremal values $R_1(r)$ and $R_2(r)$ we elaborated a programme and we used the computer FELIX C–256.

In the Table 1 are presented some of the results obtained by running the programme for various values of r :

Nr.	r	$r/(1+r)^2$	$R_1(r)$	$R_2(r)$	$r/(1-r)^2$
1.	0,664152	0,2398	0,6107	1,9493	5,8881
2.	0,727453	0,2437	0,6472	2,0298	9,7931
3.	0,748531	0,2448	0,6533	3,8312	11,8369

Table 1.

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