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ON THE DIVERGENCE OF LAGRANGE INTERPOLATION PROCESSES

by

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The paper deals with some aspects of the divergence of Lagrange interpolation processes. A series of studies on this problem are quite old, most of the results are classical by now. In the last years, several new results and theorems have been obtained. They have been favorised by the theorem descovered by P. Erdős [2]. A new proof of this theorem was recently published in [3]. Among the most important works on this subject, the results obtained by S. S. Pilipčuk [4], [5], [6], [7], A. A. Privalov [9], [10], L. P. Povčun [8] must be mentioned. A result on the same topic was obtained by I. Muntean and S. Cobzaş [1], using a principle of a double condensation of singularities.

In this paper, a result related to the theorem of S. S. Pilipčuk from [4] is essentially pointed out. In the first part of the paper, some properties on which the proof of the fundamental theorem is based are presented.

I. Let $C[\alpha, \beta]$ denote the set of all real continuous functions on the interval $[\alpha, \beta]$, where $\alpha, \beta \in \mathbf{R}$, $\alpha < \beta$, and let $f: [\alpha, \beta] \to \mathbf{R}$.

We define $R_f = \{x \in [\alpha, \beta] : f(x) = 0\}$. Let $a \in [\alpha, \beta]$. Denote by $G(a, \alpha, \beta)$ the set of all functions $f: [\alpha, \beta] \to \mathbf{R}$ which have the following properties:

(i) $f \in C[\alpha, \beta]$.

(ii) $R_f = \{a\}$.

(iii) There exists $\delta > 0$ so that f is strictly monotonic on the intervals $[a - \delta, a] \cap [\alpha, \beta]$ and $[a, a + \delta] \cap [\alpha, \beta]$.

Remark. If $f \in G(a, \alpha, \beta)$, then $|f| \in G(a, \alpha, \beta)$.

The following lemma is valid:

LEMMA. *If* $h \in G(0, 0, 1)$, *then* $\lim\limits_{t\to 0} h^2(t)\int_t^1 (h(\tau))^{-1}\, d\tau = 0.$

Proof. Since $R_h = \{0\}$ and h is continuous, we have either $h(x) > 0$, or $h(x) < 0$, $\forall x \in\ ]0, 1]$. Consider the first situation. From the hypothesis of Lemma, it follows that there is a point $\delta \in\ ]0, 1]$ so that h is a strictly increasing function on $[0, \delta]$. There exists two points $\xi \in\ ]t, \delta[$ and $\eta \in\ ]\delta, 1[$ (for $\delta < 1$) so that $0 < h^2(t)\int_t^1 (h(\tau))^{-1}\, d\tau = h^2(t)\int_t^\delta (h(\tau))^{-1} d\tau + h^2(t)\int_\delta^1 (h(\tau))^{-1}\, d\tau = h(t)\left[\frac{h(t)}{h(\xi)}(\delta - t) + (1 - \delta)\frac{h(t)}{h(\eta)}\right] < h(t)\left[\delta - t + (1 - \delta)\frac{h(t)}{h(\eta)}\right].$

Letting $t \to 0$, the conclusion follows.

Remark 1. The condition (iii) from the definition of $G(0, 0, 1)$ is essential for the conclusion of Lemma.

Consider $g :]0, 1] \to \mathbf{R}$, $g\left(\frac{1}{n}\right) = n$, $\forall n \in \mathbf{N}$, $g\left(\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n-1}\right)\right) = n^3$, $\forall n \in \mathbf{N}$, $n \geq 2$ and g linear on the intervals $\left[\frac{1}{n}, \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n-1}\right)\right]$ and $\left[\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n-1}\right), \frac{1}{n-1}\right]$, $\forall n \in \mathbf{N}$ $n \geq 2$. The function h: $[0, 1] \to \mathbf{R}$, $h(0) = 0$, $h(x) = (g(x))^{-1}$, $x \in\ ]0, 1]$, has the following properties: (1°) $R_h = \{0\}$; (2°) $h \in C[0, 1]$; (3°) h is not monotonic on any interval $[0, \delta]$, $0 < \delta \leq 1$; (4°) $\lim\limits_{t\to 0} h^2(t) \cdot \int_t^1 (h(\tau))^{-1}\, d\tau \neq 0.$

Indeed, (1°) is obvious. In order to prove (2°), it suffices to show that h is continuous in $x_0 = 0$ (since g is continuous). Let (x_n) be a sequence of numbers such that $x_n \to 0$ and $x_n > 0$. There exists $k_n \in \mathbf{N}$, $k_n \geq 2$ so that $x_n \in \left[\frac{1}{k_n}, \frac{1}{k_n - 1}\right]$. We have $g(x_n) \geq g\left(\frac{1}{k_n - 1}\right)$, hence $0 < h(x_n) < h\left(\frac{1}{k_n - 1}\right) = \frac{1}{k_n - 1}$. Since $x_n \to 0$, we have $k_n \to \infty$ and thus $h(x_n) \to 0$.

To prove (3°), suppose (3°) is false. Then there exists some $\delta_0 \in\ ]0, 1]$, such that suh is monotonic on $[0, \delta_0]$. Since $h(0) = 0$ and $h(\delta_0) > 0$, it follows that h this increasing on $[0, \delta_0]$. Let $n_0 \in \mathbf{N}$ so that $(n_0 - 1)^{-1} < \delta_0$. From $\frac{1}{n_0} < \frac{1}{2}\left(\frac{1}{n_0} + \frac{1}{n_0 - 1}\right) < \frac{1}{n_0 - 1} < \delta_0$, we have $h\left(\frac{1}{n_0}\right) < h\left(\frac{1}{2}\left(\frac{1}{n_0} + \frac{1}{n_0 - 1}\right)\right)$ or $\frac{1}{n_0} < \frac{1}{n_0^3}$, which is a contradiction.

Finally, let's show that $\lim\limits_{t\to 0} h^2(t)\int_t^1 (h(\tau))^{-1}\, d\tau \neq 0.$

It is enough to exist a sequence (x_n), $x_n \in\ ]0, 1]$ so that $x_n \to 0$ and

$$\lim_{n\to\infty} h^2(x_n)\int_{x_n}^1 (h(\tau))^{-1} d\tau \neq 0.$$

Taking $x_n = \frac{1}{n}$, we have

$$\int_{x_n}^1 (h(\tau))^{-1}\, d\tau = \sum_{k=2}^{n}\left[\int_{\frac{1}{k}}^{\frac{1}{2}\left(\frac{1}{k}+\frac{1}{k-1}\right)} g(\tau)\, d\tau + \int_{\frac{1}{2}\left(\frac{1}{k}+\frac{1}{k-1}\right)}^{\frac{1}{k-1}} g(\tau)\, d\tau\right].$$

Since $g(t) = 2n^2(n-1)^2(n+1)t - 2n(n-1)^2(n+1) + n$, for $t \in \left[\frac{1}{n}, \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n-1}\right)\right]$ and $g(t) = -2n^2(n-1)^2(n-2)t + 2n^2(n-1)(n-2) + n - 1$, for $t \in \left[\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n-1}\right), \frac{1}{n-1}\right]$, $n \geq 2$, we obtain:

$$\int_{x_n}^1 (h(\tau))^{-1}\, d\tau = \sum_{k=2}^{n} a_k, \text{ where } a_k = \frac{1}{2}k - \frac{1}{4} + \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k-1}\right).$$

Then,

$$\lim_{n\to\infty} h^2(x_n)\int_{x_n}^1 (h(\tau))^{-1}\, d\tau = \lim_{n\to\infty}\frac{1}{n^2}\sum_{k=2}^{n} a_k = \lim_{n\to\infty}\frac{a_{n+1}}{2n+1} = \frac{1}{4} \neq 0.$$

Remark 2. The condition (iii) from the definition of $G(0, 0, 1)$ is not necessary for the conclusion of Lemma.

Indeed, the function h: $[0, 1] \to \mathbf{R}$, $h(0) = 0$, $h\left(\frac{1}{n}\right) = \frac{1}{n}$, $n \in \mathbf{N}$, $h\left(\frac{1}{2}\left(\frac{1}{n} + \frac{1}{n-1}\right)\right) = \frac{1}{2n}$, $n \in \mathbf{N}$, $n \geq 2$ and h linear elsewhere, has the properties:

(1°) $R_h = \{0\}$;

(2°) $h \in C\ [0, 1]$;

(3°) h is not monotonic on any interval $[0, \delta]$, where $\delta \in\ ]0, 1]$;

(4°) $\lim\limits_{t\to 0} h^2(t)\int_t^1 (h(\tau))^{-1}\, d\tau = 0.$

It is clear that (1°) holds. To prove (2°) it is enough to show that h is continuous in $x = 0$. Let $x_n \to 0$, $x_n > 0$. There exists $k_n \in \mathbf{N}$, $k_n \geqslant 2$ such that $x_n \in \left[\frac{1}{k_n}, \frac{1}{k_n - 1}\right]$. We have $0 \leqslant h(x_n) \leqslant h\left(\frac{1}{k_n - 1}\right) = \frac{1}{k_n - 1}$. Since $x_n \to 0$, it follows that $k_n \to \infty$ and thus $h(x_n) \to 0$, as $n \to \infty$ which shows that h is continuous in $x = 0$.

In order to prove (3°), let's suppose that there exists $\delta_0 \in]0, 1]$ so that h is monotonic on $[0, \delta_0]$. Since $h(\delta_0) > 0$, it follows that h is increasing on $[0, \delta_0]$. Choose $n_0 \in \mathbf{N}$ so that $(n_0 - 1)^{-1} < \delta_0$. Since $\frac{1}{n_0} < \frac{1}{2}\left(\frac{1}{n_0} + \frac{1}{n_0 - 1}\right) < \frac{1}{n_0 - 1} < \delta_0$, we have $h\left(\frac{1}{n_0}\right) < h\left(\frac{1}{2}\left(\frac{1}{n_0} + \frac{1}{n_0 - 1}\right)\right)$ or $\frac{1}{n_0} < \frac{1}{2n_0}$, a contradiction.

To prove (4°), let $x_n \to 0$, $x_n > 0$. There exists $k_n \in \mathbf{N}$, $k_n \geqslant 2$ such that $x_n \in \left[\frac{1}{k_n}, \frac{1}{k_n - 1}\right]$. We have

$$h^2(x_n)\int_{x_n}^{1} (h(\tau))^{-1}\, d\tau \leqslant h^2(x_n)\int_{\frac{1}{k_n}}^{1} (h(\tau))^{-1}\, d\tau = h^2(x_n)\sum_{j=2}^{k_n}\int_{\frac{1}{j}}^{\frac{1}{j-1}} (h(\tau))^{-1}\, d\tau.$$

Since $h(t) \geqslant \frac{1}{2j}$, $\forall t \in \left[\frac{1}{j}, \frac{1}{j - 1}\right]$, we obtain

$$0 \leqslant h^2(x_n)\cdot\int_{x_n}^{1} (h(\tau))^{-1}\, d\tau \leqslant h^2(x_n)\sum_{j=2}^{k_n} 2j\left(\frac{1}{j-1} - \frac{1}{j}\right) \leqslant \frac{1}{(k_n - 1)^2}\sum_{j=2}^{k_n}\frac{2}{j-1} \leqslant$$

$$\leqslant \frac{2(k_n - 1)}{(k_n - 1)^2} = \frac{2}{k_n - 1}.$$

Letting $n \to \infty$, (4°) follows, since $k_n \to \infty$.

Remark 3. The condition (ii) may be replaced by the condition: (ii') The set R_f is not empty and contains a finite number of elements, then formulating for each $a \in R_f$ one condition of the type (iii).

Remark 4. If the hypothesis of Lemma holds, it can be proved that

$$\lim_{t\to 0} h^{\alpha}(t)\cdot\int_{t}^{1} (h(\tau))^{-1} d\tau = 0,\quad \forall \alpha > 1;\ \text{for } \alpha = 1, \text{ the statement is false.}$$

COROLLARY 1. *If* $g \in G(x_0, -1, 1)$, *then:*

$$\lim_{\substack{x\to x_0\\ x<x_0}} g^2(x)\cdot\int_{-1}^{x} |g(t)|^{-1}\, dt = 0,\ \textit{for}\ x_0 \neq -1\ \textit{and}$$

$$\lim_{\substack{x\to x_0\\ x>x_0}} g^2(x)\cdot\int_{x}^{1} |g(t)|^{-1}\, dt = 0,\ \textit{for}\ x_0 \neq 1.$$

Proof. In the first case, we define $s : [0, 1] \to [-1, x_0]$, $s(t) = (-1 - x_0)t + x_0$. The function $h = |g| \circ s$ satisfies the hypothesis of Lemma and thus

$$\lim_{t\to\infty} h^2(t)\int_{t}^{1} (h(\tau))^{-1}\, d\tau = 0.$$

The change of variables on the integral defined by $\tau = s^{-1}(u)$, implies the first equality, where $x = s(t)$.

In the second case, we define $s : [0, 1] \to [x_0, 1]$, $s(t) = (1 - x_0)t + x_0$ and $h = |g| \circ s$.

COROLLARY 2. *If* $g \in G(x_0, -1, 1)$ *and* $x_0 \neq \pm 1$, *then*

$$\lim_{\varepsilon\to 0,\ \varepsilon>0} g^2(x_0 - \varepsilon)\int_{-1}^{x_0-\varepsilon} |g(t)|^{-1}\, dt = 0,\quad \lim_{\varepsilon\to 0,\ \varepsilon>0} g^2(x_0 + \varepsilon)\int_{x_0+\varepsilon}^{1} |g(t)|^{-1}\, dt = 0$$

and

$$\lim_{\varepsilon\to 0,\ \varepsilon>0}\int_{x_0-\varepsilon}^{x_0+\varepsilon} |g(t)|dt = 0.$$

Proof. The first two equalities follow from Corollary 1, putting $x_0 - x = \varepsilon$, respectively $x - x_0 = \varepsilon$, and the last one from the relation:

$$0 < \int_{x_0-\varepsilon}^{x_0+\varepsilon} |g(t)|dt \leqslant 2\varepsilon\cdot\max\{|g(x)| : x \in [-1, 1]\}.$$

II. Let $\mathfrak{M}$ be a triangular infinite matrix of nodes in the interval $[-1, 1]$, $-1 \leqslant x_n^1 < x_n^2 < x_n^3 < \ldots < x_n^n \leqslant 1$, $n = 1, 2, 3, \ldots$. Let $f : [-1, 1] \to \mathbf{R}$. We put $\omega_n(x) = \prod_{k=1}^{n} (x - x_n^k)$, $l_n^k(\mathfrak{M}; x) = \omega_n(x)[\omega_n'(x_n^k)(x - x_n^k)]^{-1}$,

$$L_n(\mathfrak{M}; f; x) = \sum_{k=1}^{n} f(x_k^n)\cdot l_n^k(\mathfrak{M}; x),$$

$$L_n(\mathfrak{M}; x) = \sum_{k=1}^{n} |l_n^k(\mathfrak{M}; x)|,\quad d_n = \min\{|x_n^{k+1} - x_n^k| : 1 \leqslant k \leqslant n - 1\}.$$

Since $\sum_{k=1}^{n} l_n^k(\mathfrak{M}; x) = 1$, $\forall n \in \mathbf{N}$, $\forall x \in [-1, 1]$, we have:

$$L_n(\mathfrak{M}; x) \geqslant 1,\quad \forall n = 1, 2, 3, \ldots,\quad \forall x \in [-1, 1]. \tag{1}$$

In [4], S. S. Pilipčuk shows that there is a set $E = E(\mathfrak{M})$, $E \subset [-1,1]$, $m(E) = 2$, so that for every point $x_0 \in E$, there exists a function $f \in C[-1, 1]$ which has the properties:

$$f(x_0) = 0, \tag{2}$$

$$\int_{-1}^{1} |f(x)(x - x_0)^{-1}| \, dx < \infty, \tag{3}$$

and

$$\limsup_{n \to \infty} |L_n(\mathfrak{M}; f; x_0)| = +\infty. \tag{4}$$

In the same paper, the following remark has been made: for matricies $\mathfrak{M}$ which have the property that for infinite sequences (n_i) and for every $\varepsilon \in \left]0, \frac{2}{3}\right[$, the following inequality holds:

$\max \{|l_n^k(\mathfrak{M}, x)| : -1 + \varepsilon \leqslant x \leqslant 1 - \varepsilon\} \leqslant C$, $1 \leqslant k \leqslant n_i$, $i = 1, 2, 3, \ldots$, there exists a set $E = E(\mathfrak{M})$, $E \subset [-1, 1]$, $m(E) > 2 - 3\varepsilon$ so that for every point $x_0 \in E$ there is a nonnegative function $f \in C[-1, 1]$, which satisfies (2), (3), (4).

In connection with these statements, we will prove the following theorem:

THEOREM *If $\mathfrak{M}$ is a triangular infinite matrix of nodes in the interval $[-1, 1]$, then there exists a set $E \subset]-1, 1[$ with $m(E) = 2$, so that for every $x_0 \in E$ and for every function $g \in G(x_0, -1, 1)$, there exists a function $f: [-1, 1] \to \mathbf{R}$ which satisfies the following properties:*

(1°) $f(x_0) = 0$

(2°) $f(x) > 0$, $\forall x \in [-1, 1]$, $x \neq x_0$,

(3°) $f \in C[-1, 1]$,

(4°) $\int_{-1}^{1} f(x) \cdot |g(x)|^{-1} \, dx < \infty$,

(5°) $\limsup_{n \to \infty} |L_n(\mathfrak{M}; f; x_0)| = +\infty$.

Proof. We shall divide the proof into five steps.

(II.1) There exists a set $E \subset]-1, 1[$ with $m(E) = 2$ so that for every $x \in E$ we have $\limsup_{n \to \infty} L_n(\mathfrak{M}; x) = +\infty$ (see [2] and [3]).

Let $x_0 \in E$. It is easily seen that there is a subsequence $(L_{n_i}(\mathfrak{M}; x_0))$ of $(L_n(\mathfrak{M}; x_0))$ so that:

(1.*i*) $L_{n_{i_1}}(\mathfrak{M}; x_0) < L_{n_{i_2}}(\mathfrak{M}; x_0)$, $\forall i_1 < i_2$,

(1.*ii*) $L_{n_i}(\mathfrak{M}; x_0) > (4i + 1)^2$, $\forall i \in \mathbf{N}$,

(1.*iii*) $\sum_{i=1}^{\infty} [L_{n_i}(\mathfrak{M}; x_0)]^{-\frac{1}{2}} < 1$

(1.*iv*) $L_{n_i}(\mathfrak{M}; x_0) \sum_{k=i+1}^{\infty} [L_{n_k}(\mathfrak{M}; x_0)]^{-\frac{1}{2}} < 1$, $\forall i \in \mathbf{N}$.

(II.2) Let n_i be given. Since $x_0 \neq x_{n_i}^k$, $\forall k \in \{1, \ldots, n_i\}$ (otherwise (1.*iii*) will be false), there exists $p_i \in \mathbf{N}$ such that $x_{n_i}^{p_i} < x_0 < x_{n_i}^{p_i+1}$ (we put $x_{n_i}^0 = -1$, if $x_{n_i}^1 > -1$, and $x_{n_i}^{n_i+1} = 1$, if $x_{n_i}^{n_i} < 1$).

Let $g \in G(x_0, -1, 1)$ and $\beta_i = \min\{|g(x_{n_i}^k)| : 1 \leqslant k \leqslant n_i\}$.

By the continuity of g, there is a $\delta_i' > 0$, so that $\forall x \in [x_{n_i}^k - \delta_i', x_{n_i}^k + \delta_i']$, $1 \leqslant k \leqslant n_i$, we have $|g(x)| > 2^{-1} \cdot \beta_i$.

Put $\delta_i = 2^{-1} \min \{d_{n_i}, x_0 - x_{n_i}^{p_i}, x_{n_i}^{p_i+1} - x_0, \delta_i', (2n_i)^{-1} \cdot \beta_i\}$.

Then we have:

$$\forall x \in [x_{n_i}^k - \delta_i, x_{n_i}^k + \delta_i], \ 1 \leqslant k \leqslant n_i : |g(x)|^{-1} \leqslant 2\beta_i^{-1} \tag{2.1}$$

$$4n_i \, \delta_i \, \beta_i^{-1} \leqslant 1. \tag{2.2}$$

By Corollary 2 (of I), by the continuity of g at x_0 and by the fact that $g(x_0) = 0$, there exists $\varepsilon_i \in \mathbf{R}$ so that:

$$\begin{cases} \varepsilon_i > 0, \ x_0 - \varepsilon_i \geqslant x_{n_i}^{p_i} + \delta_i, \ x_0 + \varepsilon_i \leqslant x_{n_i}^{p_i} - \delta_i, \\ g^2(x_0 - \varepsilon_i) < [L_{n_i}(\mathfrak{M}; x_0)]^{-1}, \ g^2(x_0 + \varepsilon_i) < [L_{n_i}(\mathfrak{M}; x_0)]^{-1}, \\ g^2(x_0 - \varepsilon_i) \int_{-1}^{x_0 - \varepsilon_i} |g(t)|^{-1} \, dt \leqslant 3^{-1} \cdot 2^{-i}, \\ g^2(x_0 + \varepsilon_i) \int_{x_0 + \varepsilon_i}^{1} |g(t)|^{-1} dt \leqslant 3^{-1} \cdot 2^{-i}, \ \int_{x_0 - \varepsilon_i}^{x_0 + \varepsilon_i} |g(t)| \, dt \leqslant 3^{-1} \cdot 2^{-i}. \end{cases} \tag{2.3}$$

Let

$A_1 = \{x_{n_i}^k / l_{n_i}^k(\mathfrak{M}; x_0) < 0, \ 1 \leqslant k \leqslant p_i\}$, for $p_i \geqslant 1$

$A_2 = \{x_{n_i}^k / l_{n_i}^k(\mathfrak{M}; x_0) < 0, \ p_i + 1 \leqslant k \leqslant n_i\}$, for $p_i \leqslant n_i - 1$

$A_3 = \{x_{n_i}^k / l_{n_i}^k(\mathfrak{M}; x_0) > 0, \ 1 \leqslant k \leqslant p_i\}$, for $p_i \geqslant 1$

$A_4 = \{x_{n_i}^k / l_{n_i}^k(\mathfrak{M}; x_0) > 0, \ p_i + 1 \leqslant k \leqslant n_i\}$, for $p_i \leqslant n_i - 1$.

(II.3) Next, we define the functions $\varphi_{i1} : [-1,\ 1] \to \mathbf{R}$,

$\varphi_{i2} : [-1,\ 1] \to R,\ \varphi_i : [-1,\ 1] \to \mathbf{R}$,

$$\varphi_{i1}(x) = \begin{cases} g^2(x_0 - \varepsilon_i), & \text{for } x \in [-1,\ x_0 - \varepsilon_i[\\ g^2(x), & \text{for } x \in [x_0 - \varepsilon_i,\ x_0 + \varepsilon_i] \\ g^2(x_0 + \varepsilon_i), & \text{for } x \in [x_0 + \varepsilon_i,\ 1.] \end{cases}$$

$$\varphi_{i2}(x) = \begin{cases} 0, \quad \text{for } x \in [-1,\ 1] \setminus \bigcup_{k=1}^{n_i} [x_{n_i}^k - \delta_i,\ x_{n_i}^k + \delta_i] \\ [L_{n_i}(\mathfrak{M}\ ;\ x_0)]^{-1} - g^2(x_0 - \varepsilon_i), \ \text{for } x \in A_1 \\ [L_{n_i}(\mathfrak{M}\ ;\ x_0)]^{-1} - g^2(x_0 + \varepsilon_i), \ \text{for } x \in A_2 \\ [L_{n_i}(\mathfrak{M}\ ;\ x_0)]^{-\frac{1}{2}} - g^2(x_0 - \varepsilon_i), \ \text{for } x \in A_3 \\ [L_{n_i}(\mathfrak{M}\ ;\ x_0)]^{-\frac{1}{2}} - g^2(x_0 + \varepsilon_i), \ \text{for } x \in A_4 \\ \text{linear}, \quad \text{for the rest} \end{cases}$$

$$\varphi_i(x) = \varphi_{i1}(x) + \varphi_{i2}(x), \qquad \forall x \in [-1,\ 1].$$

We have:

(3.1) $\varphi_i(x_0) = 0,$

(3.2) $\varphi_i \in C[-1,\ 1]$

(3.3) $\varphi_i(x) > 0,\ \forall x \in [-1,\ 1],\ x \neq x_0,$

(3.4) $\max\ \{\varphi_i(x) : x \in [-1,\ 1]\} = [L_{n_i}(\mathfrak{M}\ ;\ x_0)]^{-\frac{1}{2}},$

(3.5) $\int_{-1}^{1} \varphi_i(x) \cdot |g(x)|^{-1}\, dx \leqslant 2^{-i} + [L_{n_i}(\mathfrak{M}\ ;\ x_0)]^{-\frac{1}{2}}.$

It is clear that (3.1) and (3.2) hold. Next, (3.3) and (3.4) follow from (1), (2.3) and the definition of φ_i.

Let us establish (3.5). We have:

$$\int_{-1}^{1} \varphi_i(x)\, |g(x)|^{-1}\, dx = \int_{-1}^{1} \varphi_{i1}(x)\, |g(x)|^{-1}\, dx +$$

$$+ \int_{-1}^{1} \varphi_{i2}(x)\, |g(x)|^{-1}\, dx = \int_{-1}^{x_0 - \varepsilon_i} g^2(x_0 - \varepsilon_i)\, |g(x)|^{-1}\, dx +$$

$$+ \int_{x_0 - \varepsilon_i}^{x_0 + \varepsilon_i} |g(x)|\, dx + \int_{x_0 + \varepsilon_i}^{1} g^2(x_0 + \varepsilon_i)\, |g(x)|^{-1}\, dx +$$

$$+ \sum_{k=1}^{n_i} \int_{x_{n_i}^k - \delta_i}^{x_{n_i}^k + \delta_i} \varphi_{i2}(x)\, |g(x)|^{-1}\, dx = g^2(x_0 - \varepsilon_i) \int_{-1}^{x_0 - \varepsilon_i} |g(x)|^{-1}\, dx +$$

$$+ \int_{x_0 - \varepsilon_i}^{x_0 + \varepsilon_i} |g(x)|\, dx + g^2(x_0 + \varepsilon_i) \int_{x_0 + \varepsilon_i}^{1} |g(x)|^{-1}\, dx +$$

$$+ \sum_{k=1}^{n_i} \int_{x_{n_i}^k - \delta_i}^{x_{n_i}^k + \delta_i} \varphi_{i2}(x)\, |g(x)|^{-1}\, dx.$$

By (2.1), (2.2), (2.3), (3.4) and $\varphi_{i2}(x) \leqslant \varphi_i(x),\ \forall x \in [-1,\ 1]$, we have:

$$\int_{-1}^{1} \varphi_i(x) |g(x)|^{-1}\, dx \leqslant 3^{-1} \cdot 2^{-i} + 3^{-1} \cdot 2^{-i} + 3^{-1} \cdot 2^{-i} +$$

$$+ [L_{n_i}(\mathfrak{M}\ ;\ x_0)]^{-\frac{1}{2}} \cdot \sum_{k=1}^{n_i} 2 \cdot \beta_i^{-1} \cdot 2\delta_i =$$

$$= 2^{-i} + [L_{n_i}(\mathfrak{M}\ ;\ x_0)]^{-\frac{1}{2}} \cdot 4n_i\, \delta_i \cdot \beta_i^{-1} \leqslant 2^{-i} + [L_{n_i}(\mathfrak{M}\ ;\ x_0)]^{-\frac{1}{2}}.$$

(II.4) Next, we will construct, similarly as in [4], the function f. Let us consider the sequence (n_i). We put $n_{i_1} = n_2$ and we suppose that the numbers $n_{i_1},\ n_{i_2},\ \ldots,\ n_{i_{m-1}}$ are determinated.

We define:

(4.1) $F_{m-1} : [-1,\ 1] \to \mathbf{R},\ F_{m-1}(x) = \sum_{j=1}^{m-1} \varphi_{i_j}(x),\ \forall x \in [-1,\ 1].$

If $\limsup_{n\to\infty} |L_n(\mathfrak{M};\ F_{m-1}\ ;\ x_0)| = +\infty$, then we put $n_{i_m} = 0,\ f = F_{m-1}$ and Theorem is proved.

Suppose that we have:

(4.2) $\limsup_{n\to\infty} |L_n(\mathfrak{M}\ ;\ F_{m-1}\ ;\ x_0)| = C_m < \infty.$

Then,

$$L_{n_i}(\mathfrak{M};\ \varphi_i;\ x_0) = \sum_{k=1}^{n_i} \varphi_i(x_{n_i}^k) l_{n_i}^k(\mathfrak{M};\ x_0) = \tag{4.3}$$

$$= \Sigma_+ \varphi_i(x_{n_i}^k) l_{n_i}^k(\mathfrak{M};\ x_0) + \Sigma_- \varphi_i(x_{n_i}^k) l_{n_i}^k(\mathfrak{M};\ x_0)$$

$\forall i \in \mathbf{N}$, where Σ_+ refers to the terms of sum for which $l_{n_i}^k(\mathfrak{M};\ x_0) > 0$ and Σ_- refers to the terms of sum for which $l_{n_i}^k(\mathfrak{M};\ x_0) < 0$. It is easily seen that both sums contain effectively the terms.

Now, we put

$$S_+^i = \sum_{k=1,\ l_{n_i}^k(\mathfrak{M};\ x_0) > 0}^{n_i} l_{n_i}^k(\mathfrak{M};\ x_0),$$

$$S_-^i = \sum_{k=1,\ l_{n_i}^k(\mathfrak{M};\ x_0) < 0}^{n_i} l_{n_i}^k(\mathfrak{M};\ x_0).$$

Since

$$S_+^i + S_-^i = \sum_{k=1}^{n_i} l_{n_i}^k(\mathfrak{M};\ x_0) = 1$$

and

$$S_+^i - S_-^i = \sum_{k=1}^{n_i} |l_{n_i}^k(\mathfrak{M};\ x_0)| = L_{n_i}(\mathfrak{M};\ x_0),$$

we have

$$S_+^i = 2^{-1}[1 + L_{n_i}(\mathfrak{M};\ x_0)],\ S_-^i = 2^{-1}[1 - L_{n_i}(\mathfrak{M};\ x_0)].$$

By (4.3), We obtain, using (1.*ii*):

$$L_{n_i}(\mathfrak{M};\ \varphi_i;\ x_0) = 2^{-1}[(L_{n_i}(\mathfrak{M};\ x_0))^{-\frac{1}{2}} + (L_{n_i}(\mathfrak{M};\ x_0))^{\frac{1}{2}}(L_{n_i}+(\mathfrak{M};\ x_0))^{-1} - 1] \geqslant$$

$$\geqslant 2^{-1}[(L_{n_i}(\mathfrak{M};\ x_0))^{\frac{1}{2}} - 1] \geqslant 2i.$$

There exists a number $i \in \mathbf{N}$, which we shall denote by i_m, such that $i \geqslant C_m$ and $i > i_{m+1}$.

So, we have choosen n_{i_m} and we have:

$$L_{n_{i_m}}(\mathfrak{M};\ \varphi_{i_m};\ x_0) \geqslant i_m + C_m. \tag{4.4}$$

Next, suppose that for all the numbers m we have (4.2). Then, we define:

$$f: [-1,\ 1] \to \mathbf{R},\ f(x) = \sum_{j=1}^{\infty} \varphi_{i_j}(x),\ \forall x \in [-1,\ 1]. \tag{4.5}$$

By (3.4) and (1.*iii*), it follows that this definition is correct.

(II.5) Finally, let us show that this function f has the desired properties (1°)–(5°).

The properties (1°) and (2°) follow from the definition of f, (3.1) and (3.3).

The achievement of (3°) is proved similarly to that in [4], using (1.*iii*), (3.4) and (3.2).

Refering to (4°), using the respective arguments from [4] and the relations (3.5) and (1.*iii*) we obtain

$$\int_{-1}^{1} f(x) \cdot |g(x)|^{-1}\, dx \leqslant 2 < \infty.$$

The property (5°) follows from the linearity of $L_{n_{i_m}}$ and the relations (4.4), (4.5), (4.2), (1.*iv*), (3.4). More precisely, it is shown as in [4] tha $|L_{n_{i_m}}(\mathfrak{M};\ f;\ x_0)| > i_m - 1$. Now, letting $m \to \infty$, we obtain $\limsup_{n\to\infty} L_n|(\mathfrak{M}$ $f;\ x_0)| = +\infty$, which completes the proof.

Remark. A similar argument, but simpler, shows that under the hypothesis of Theorem, there is a set $E \subset]-1,\ 1[$, $m(E) < 2$, so that $\forall x_0 \in E$, $\forall g \in G(x_0, -1,\ 1)$ $\exists f \in C[-1,\ 1]$, which satisfies the properties (1°), (5°) and

$$\int_{-1}^{1} |f(x)| \cdot |g(x)|^{-1}\, dx < \infty.$$

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