MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 13, N° 2, 1984, pp. 155 — 161

A TRIANGLE INTERPOLANT WITH LINEAR COEFFICIENTS BASED UPON THE FUNDAMENTAL BILINEAR INTERPOLANT OF MANGERON

by

GREGORY M. NIELSON
(Tempe)

1. Introduction

The fundamental bilinear interpolant of Mangeron [2],

$$(1.1)\quad M[f](x, y) = (1-y)f(x, 0) + yf(x, 1) + (1-x)f(0, y) + xf(1, y) - (1-x)(1-y)f(0, 0) - x(1-y)f(1, 0) - y(1-x)f(0, 1) - xyf(1, 1),$$

which interpolates to f on the boundary of $R = \{(x, y): 0 \leqslant x \leqslant 1;\ 0 \leqslant y \leqslant 1\}$, has had wide spread influence in the area of approximation of bivariate functions. While $M[f]$ and its generalization to include interpolation to normal derivatives serves well for a rectangular domain, many applications require approximations which are not inherently rectangular and consequently there is interest in developing analogous methods for other domains. Of particular interest, due to the application in finite element analysis and scattered data interpolation, is a triangular domain. In this report, we present methods for a triangular domain which are patterned after the following two equivalent properties that characterize $M[f]$.

Characterization 1. $M[f]$ is the unique function in $C^{2,2}(R)$ which lies in the kernel of the operator $\partial^4/\partial x^2\partial y^2$ and interpolates to f on the boundary of R.

Characterization 2. Among all functions in $C^{2,2}(R)$ which interpolate to f on the boundary of R, $M[f]$ uniquely minimizes the pseudonorm

$$\sqrt{\iint_R [h_{xy}(s, t)]^2\, ds dt}$$

2. Linear Interpolation for Triangles.

For notational convenience, we will use the standard triangular domain T with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$. Any other triangular domain can be mapped linearly to T by the use of an affine transformation. By $C^{n,m}(T)$, we mean those functions which possess continuous partial derivatives $\partial^{i+j} f/\partial x^i \partial y^j$; $i \leqslant n$, $j \leqslant m$ on T.

In the development of techniques of interpolation for triangular domains, there is a tendency towards the use of rational weight functions. This is due, in part, to the fact that M can be viewed as the boolean sum of two essentially univariate operators which are based upon linear interpolation along lines parallel to the sides of R. That is

$$M = M_x \oplus M_y = M_x + M_y - M_x M_y$$

where

$$M_x[f](x, y) = (1 - x)f(0, y) + xf(1, y)$$
$$M_y[f](x, y) = (1 - y)f(x, 0) + yf(x, 1).$$

The corresponding linear interpolation operators for the domain T have rational weight function. For example,

$$T_x[f](0, y) = [(1 - x - y)/(1 - y)]f(0, y) + [x/(1 - y)]f(1 - y, y)$$

is based upon linear interpolation along lines parallel to the x-axis and so it is clear that any scheme based upon the Boolean sum of these operators will have rational weight functions. We now proceed to develop a method which is void of rational weight functions.

Concerning the first type of characterization of $M[f]$, the general form of a function $k(x, y)$ which has the property that $\partial^4 k/\partial x^2 \partial y^2 = 0$ is

$$k(x, y) = g_1(x) + g_2(y) + yg_3(x) + xq_4(y)$$

for arbitrary, twice differentiable functions g_i, $i = 1, 2, 3, 4$. If we now select g_i, $i = 1, 2, 3, 4$ by forcing k to interpolate to f on the boundary of T, we obtain the equations

$$g_1(x) + g_2(0) + xg_4(0) = f(x, 0)$$
$$g_1(0) + g_2(y) + yg_3(0) = f(0, y)$$
$$g_1(x) + g_2(1 - x) + (1 - x)g_3(x) + xg_4(1 - x) = f(x, 1 - x).$$

Solving these equations, leads to

$$k[f](x, y) = \frac{2 - y - 2x}{2(1 - x)} f(x, 0) + \frac{2 - 2y - x}{2(1 - y)} f(0, y) + \frac{(x + y)(3 - x - y) - 2}{2(1 - x)(1 - y)} f(0,0)$$
$$+ \frac{x}{2(1 - y)} [f(1 - y, y) - f(1 - y, 0)] + \frac{y}{2(1 - x)} [f(x, 1 - x) - f(0, 1 - x)] + \tag{2.1}$$
$$+ xy[g(x) - g(1 - y)]$$

which interpolates to f on the boundary of T for arbitrary g. In order to eliminate the rational weight functions, we choose

$$g(x) = \frac{1 - 2x}{2x(1 - x)} [f(x, 1 - x) - f(0, 1 - x) - f(x, 0) + f(0, 0)]$$

which leads to the interpolation formula

$$B[f](x, y) = (1 - y)f(x, 0) + (1 - x)f(0, y) - (1 - x - y)f(0, 0) \tag{2.2}$$
$$+ x[f(1 - y, y) - f(1 - y, 0)] + y[f(x, 1 - x) - f(0, 1 - x)].$$

It is interesting that this choice of g not only yields an interpolant with linear weights, but it also leads to an approximation which has a property analogous to the second characterization of $M[f]$.

THEOREM 2.1. *Among all functions in $C^{2,2}(T)$ which interpolate to f on the boundary of T, $B[f]$ uniquely minimizes $< \cdot, \cdot >$ where*

$$< h, g > = \iint_T h_{xy}(s, t)\, g_{xy}(s, t)\, ds\, dt.$$

Proof: First let h be any function in $C^{2,2}(T)$ which is zero on the boundary of T; then by using integration by parts we have

$$< h, B[f] > = \int_0^1 \int_0^{1-t} h_{xy}(s, t) \frac{\partial^2 B[f]}{\partial s \partial t} (s, t)\, ds dt$$

$$= \int_0^1 h_y(1 - t, t) \frac{\partial^2 B[f]}{\partial s \partial t} (1 - t, t) dt$$

$$- \int_0^1 h_y(0, t) \frac{\partial^2 B[f]}{\partial s \partial t} (0, t) dt \tag{2.3}$$

$$- \iint_T h_y(s, t) \frac{\partial^3 B[f]}{\partial s^2 \partial t} (s, t)\, ds dt$$

where

$$\frac{\partial^2 B[f]}{\partial s \partial t} (s, t) = f_x(s, 1 - s) - f_y(s, 1 - s) - f_x(s, 0) + \tag{2.4}$$
$$+ f_y(0, 1 - s) + f_y(1 - t, t) - f_x(1 - t, t) - f_y(0, t) + f_x(1 - t, 0).$$

Since $h(0, t) = 0$ we have that $h_y(0, t) = 0$ and from (2.4), we can see that $\frac{\partial^2 B[f]}{\partial s \partial t}(1 - t, t) = 0$ and so the first two terms of the right side of (2.3) vanish. Using integration by parts again, we have

$$<h, B[f]> = -\int_0^1 \int_0^{1-s} h_y(s, t) \frac{\partial^3 B[f]}{\partial s^2 \partial t}(s, t) dt ds$$

$$= -\int_0^1 h(s, 1 - s) \frac{\partial^3 B[f]}{\partial s^2 \partial t}(s, 1 - s) ds$$

$$+ \int_0^1 h(s, 0) \frac{\partial^3 B[f]}{\partial s^2 \partial t}(s, 0) ds$$

$$+ \iint_T h(s, t) \frac{\partial^4 B[f]}{\partial s^2 \partial t^2}(s, t) dt ds.$$

The first two terms are zero because $h = 0$ on the boundary of T. The last term is zero because $\partial^4 B[f]/\partial s^2 \partial t^2$ is zero on T.

Let $g \in C^{2,2}(T)$ have the property that it interpolates to f on the boundary of T. Then,

$$<g, g> - <B[f], B[f]> = <g - B[f], g - B[f]>$$
$$+ 2<g - B[f], B[f]>$$
$$= <g - B[f], g - B[f]> \geqslant 0$$

and so we have established the minimum property of $B[f]$. In order to show uniqueness, we assume the existence of another minimizing interpolant, say $\bar{g}$, and consider the error $e = \bar{g} - B[f]$. Since both minimize the pseudonorm, we have that

$$<e, e> = <\bar{g} - B[f], \bar{g} - B[f]> = 0,$$

which implies that $e_{xy} = 0$ on T. This, along with the fact that $e = 0$ on the boundary of T, implies that $e = 0$ on T which concludes the argument.

COROLLARY 2.2. The operator B is exact for any function of the form

$$g(x, y) = g_1(x) + g_2(y) + x g_3(y) + y g_3(1 - x)$$

for arbitrary functions g_i, $i = 1, 2, 3$.

Proof: Applying B directly and using

$$g(x, 0) = g_1(x) + g_2(0) + x g_3(0)$$
$$g(0, y) = g_2(y) + g_1(0) + y g_3(1)$$
$$g(0, 0) = g_1(0) + g_2(0)$$
$$g(1 - y, y) - g(1 - y, 0) = g_2(y) + g_3(y) - g_2(0) - (1 - y) g_3(0)$$
$$g(x, 1 - x) - g(0, 1 - x) = g_1(x) + g_3(1 - x) - g_1(0) - (1 - x) g_3(1)$$

will yield this result.

In order to analyze the error and rate of convergence of this approximation ,we introduce the triangular domain T_h with vertices (0, 0), (0, h) and (h, 0) and the corresponding operator

$$B_h[f](x, y) = B[f(xh, yh)]\left(\frac{x}{h}, \frac{y}{h}\right)$$

$$= \frac{1}{h}[(h - y)f(x, 0) + (h - x)f(0, y) - (h - x - y)f(0, 0) +$$
$$+ x[f(h - y, y) - f(h - y, 0)] + y[f(x, h - x) - f(0, h - x)].$$

THEOREM 2.3. *For* $f \in C^{1,1}(T_h)$

$$|B_h[f](x, y) - f(x, y)| \leqslant \frac{h^2}{2} ||f_{xy}||_h, \quad (x, y) \in T_h;$$

where the norm is the uniform on T_h.

Proof: Since it is true in general for $f \in C^{(1,1)}(T_h)$ that

$$f(x, y) - f(x, 0) - f(0, y) + f(0, 0) = \int_0^x \int_0^y f_{xy}(s, t) ds\, dt,$$

it follows that

$$B_h[f](x, y) = \frac{y}{h} \int_0^x \int_0^{h-x} f_{xy}(s, t) ds\, dt,$$

$$+ \frac{x}{h} \int_0^{h-y} \int_0^y f_{xy}(s, t) ds\, dt$$

$$- \int_0^x \int_0^y f_{xy}(s, t) ds\, dt.$$

Therefore

$$|B_h[f](x, y) - f(x, y)| \leqslant \frac{xy(h-x)}{h} ||f_{xy}||_h + \frac{x(h-y)y}{h} ||f_{xy}||_h + xy ||f_{xy}||_h$$

$$= \frac{xy(3h - x - y)}{6h} ||f_{xy}||_h.$$

This bound is maximized for $(x, y) \in T_h$ when $x = y = \frac{h}{2}$ to yield the desired result.

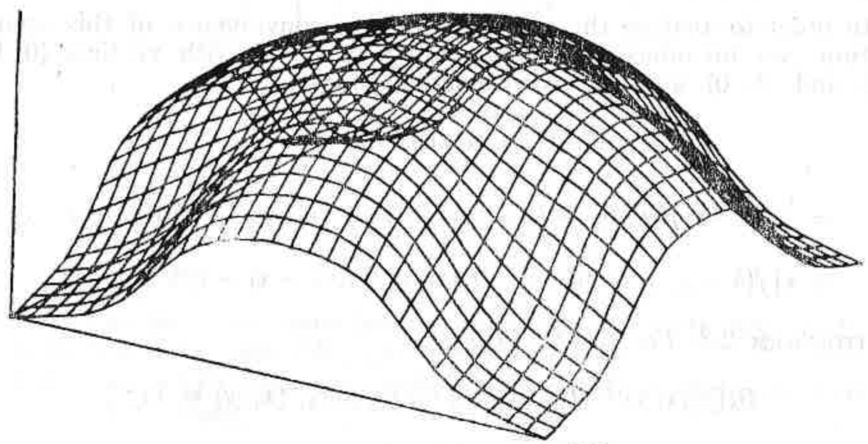

Fig. 1. These are examples of surface interpolants based upon a Triangular version of the Mangeron Interpolant.

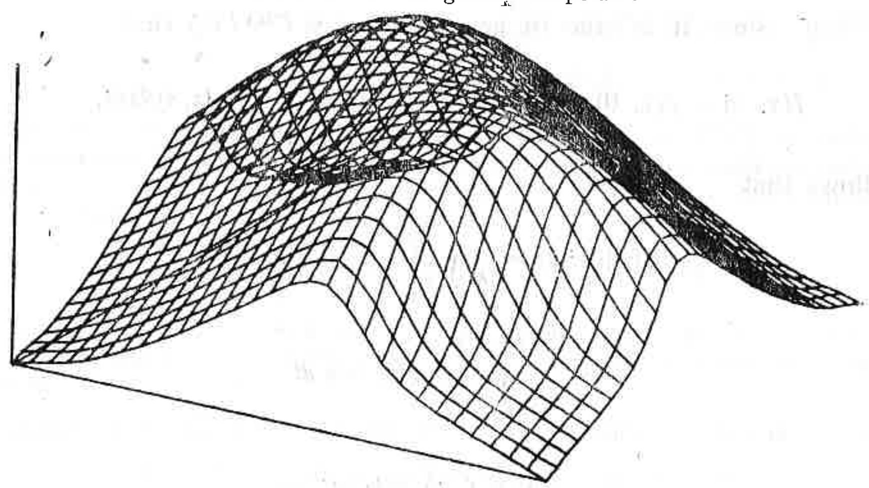

Fig. 2. These are examples of surface interpolants based upon a Triangular version of the Mangeron Interpolant.

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Received 12.I.1982.

Departament of Mathematics
Arizona, State University Tempe,
Arizona 85281