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ON THE BARYCENTER FORMULA

by

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Let E be a locally convex Hausforff space over $\mathbf{R}$ and let K be a compact convex metrizable subset of E. Let $s \in C(K)$ be a strictly convex function. If μ is a probability Radon measure on K, we denote by x_μ the barycenter of μ (see [1]). Then

$$\int h(z)\, d\mu(z) = h(x_\mu) \tag{1}$$

for every affine function $h \in C(K)$, and

$$\int g(z)\, d\mu(z) \geq g(x_\mu) \tag{2}$$

for every convex function $g \in C(K)$.

Formula (1) is referred to as the "barycenter formula".

If $f \in C(K)$, $a \in (0, 1)$ and $x, y \in K$, let us denote

$$(x,\ a,\ y\,;\ f) = (1-a)\ f(x) + af(y) - f((1-a)x + ay).$$

We need the following version of a result of T. Popoviciu (see [4]):

THEOREM 1. *Let L be a linear functional on $C(K)$ such that $L(u) > 0$ for any strictly convex function $u \in C(K)$. Then for every $f \in C(K)$ there exist $x, y \in K$, $x \neq y$ and $a \in (0, 1)$ such that:*

$$L(f) = L(s) \frac{(x,\ a,\ y\,;\ f)}{(x,\ a,\ y\,;\ s)}. \tag{3}$$

We need also:

LEMMA 1([5]). *Let L be a bounded linear functional on $C(K)$ with the supremum norm, $L \neq 0$, such that $L(g) \geq 0$ for any convex function $g \in C(K)$. Then $L(u) > 0$ for every strictly convex function $u \in C(K)$.*

Combining Theorem 1 and Lemma 1 we obtain:

THEOREM 2. *Let L be a bounded linear functional on $C(K)$, $L \neq 0$, such that $L(g) \geq 0$ for any convex function $g \in C(K)$. Then for every $f \in C(K)$ there exist $x, y \in K$, $x \neq y$ and $a \in (0, 1)$ such that (3) is satisfied.*

From (2) and Theorem 2 it follows:

COROLLARY 1. *If μ is a probability Radon measure on K, then for any $f \in C(K)$ there exist $x, y \in K$, $x \neq y$ and $a \in (0, 1)$ such that:*

$$\int f(z)\, d\mu(z) = f(x_\mu) + \left(\int s(z)\, d\mu(z) - s(x_\mu)\right) \frac{(x, a, y; f)}{(x, a, y; s)}. \tag{4}$$

If f is affine, then $(x, a, y; f) = 0$ and (4) reduces to (1).

LEMMA 2. *Let $(E, \|\cdot\|)$ be a uniformly convex space, K a compact convex subset of E and $x_0 \in K$. For any $n \in \mathbf{N}$ let μ_n be a probability Radon measure on K having x_n as barycenter. If*

$$\lim_{n\to\infty} \int h(z)\, d\mu_n(z) = h(x_0) \quad \textit{for all } h \in E', \tag{5}$$

$$\lim_{n\to\infty} \int \|z\|^2\, d\mu_n(z) = \|x_0\|^2, \tag{6}$$

then $\lim\limits_{n\to\infty} \|x_n - x_0\| = 0$.

Proof. From (1) and (5) we deduce that (x_n) converges to x_0 in the weak topology of E. From (2) and (6) it follows

$$\limsup_{n\to\infty} \|x_n\|^2 \leq \limsup_{n\to\infty} \int \|z\|^2\, d\mu_n(z) = \|x_0\|^2.$$

Thus $\limsup\limits_{n\to\infty} \|x_n\| \leq \|x_0\|$. Now the uniform convexity of E implies $\lim\limits_{n\to\infty} \|x_n - x^0\| = 0$.

Finally, let $(E, \|\cdot\|)$ be a normed space over $\mathbf{R}$; we suppose that there exists a real number $c > 0$ such that:

$$(x, a, y; \|\cdot\|^2) \geq ca(1 - a)\|x - y\|^2 \tag{7}$$

for any $x, y \in E$ and any $a \in (0, 1)$.

Remark. If (7) holds, then $c \leq 1$ and E is a uniformly convex space. (7) holds with $c = 1$ if and only if E is an inner-product space (see [2]).

Let K, x_0, μ_n, x_n be as in Lemma 2 and $d = \sup\{\|x_j\| : j = 0, 1, \ldots\}$. Let G be an open set, $K \subset G \subset E$ and $f : G \to \mathbf{R}$ a function with continuous second Fréchet derivative. Let M_i be real numbers such that $\|f^{(i)}(x)\| \leq M_i$ for any $x \in G$ and $i = 1, 2$.

From Corollary 1 we obtain after some calculations

$$\left|\int f(z)\, d\mu_n(z) - f(x_n)\right| \leq \left(\int \|z\|^2 d\mu_n(z) - \|x_n\|^2\right) \frac{M_2}{2c}.$$

So we have:

$$\left|\int f(z)\, d\mu_n(z) - f(x_0)\right| \leq \frac{M_2}{2c}\left|\int \|z\|^2\, d\mu_n(z) - \|x_0\|^2\right| + \left(M_1 + \frac{M_2 d}{c}\right)\|x_n - x_0\|. \tag{8}$$

Using (8), Lemma 2 and a Stone-Weierstrass density argument we can prove Korovkin type theorems (see A. Lupaş [3] for the case $E = \mathbf{R}$).

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