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ON SOME DISCRETE FRACTIONAL MAX-MIN PROBLEMS. APPLICATION TO MAX-MIN PROBLEMS IN GRAPHS

by

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1. Introduction

In this work we consider a parametrical procedure for solving a fractional discrete max-min problem. We shall apply this procedure to a fractional max-min problem on graphs.

2. Discrete fractional max-min problem

Let X and Y be two finite and nonvoid sets, and let f and g be two real functions defined on $X \times Y$. We suppose that $g(x, y) \neq 0$, for every $(x, y) \in X \times Y$.

The fractional discrete max-min problem under consideration is:
FD. Find

$$v = \max_{x \in X} \min_{y \in Y} \frac{f(x, y)}{g(x, y)}.$$

Let $h : X \times Y \to \mathbf{R}$ be the objective function of the FD problem, that is:

$$h(x, y) = \frac{f(x, y)}{g(x, y)}, \quad \forall (x, y) \in X \times Y. \tag{2.1}$$

Following the papers [2] or [10], we consider:

DEFINITION 2.1. *A pair* $(x', y') \in X \times X$ *is an optimal solution for the FD problem if the following conditions:*

$$(i)\ v = h(x', y') \geq \min_{y \in Y} h(x, y),\ \forall x \in X;$$

$$(ii)\ h(x', y') = \min_{y \in Y} h(x', y),$$

are satisfied.

On the FD problem we make the following hypothesis:

$$(H1) \qquad g(x, y) > 0,\ \forall (x, y) \in X \times Y.$$

3. Preliminary results

For every $t \in \mathbf{R}$, we consider the following max-min nonfractional problem:

$PA(t)$. Find

$$(3.1) \qquad F(t) = \max_{x \in X} \min_{y \in Y} (f(x, y) - t\ g(x, y)).$$

The parametrical methods which will be presented in the next sections involve the solving of the $PA(t)$ problem, for a finite sequence of values of the parameter t. Next, we will give some useful properties of the optimal value function F.

LEMMA 3.1. ([12]) *If the hypothesis H1 holds, then the function F is decreasing.*

THEOREM 3.1. *Let us suppose that the hypothesis H1 is satisfied. Then* $(x', y') \in X \times Y$ *is an optimal solution for the FD problem if and only if* (x', y') *is an optimal solution for the* $PA(t')$ *problem with* $t' = h(x', y')$.

Proof. Sufficiency: Suppose that (x', y') is an optimal solution for the $PA(t')$ problem, where $t' = h(x', y')$. This means that:

$$(3.2) \qquad f(x', y') - t'g(x', y') = \min_{y \in Y} (f(x', y) - t'g(x', y)),$$

$$(3.3) \qquad f(x', y') - t'g(x', y') \geq \min_{y \in Y} (f(x, y) - t'g(x, y)),\ \forall x \in X.$$

Since

$$f(x', y') - t'g(x', y') = f(x', y') - \frac{f(x', y')}{g(x', y')} \cdot g(x', y') = 0$$

from (3.2) and (3.3), it results:

$$(3.4) \qquad \min_{y \in Y} (f(x', y) - t' \cdot g(x', y)) = 0,$$

$$(3.5) \qquad \min_{y \in Y} (f(x, y) - t' \cdot g(x, y)) \leq 0,\ \forall x \in X.$$

But (3.4) is equivalent to:

$$(3.6) \qquad f(x', y) - t' \cdot g(x', y) \geq 0,\ \forall y \in Y,$$

and (3.5) is equivalent to the fact that for every $x \in X$, there exists $y_x \in Y$, such that:

$$(3.7) \qquad f(x', y_x) - t' \cdot g(x', y_x) \leq 0.$$

By $H1$, from (3.6) and (3.7), it follows:

$$(3.8) \qquad \frac{f(x', y)}{g(x', y)} \geq t' = \frac{f(x', y')}{g(x', y')},\ \forall y \in Y,$$

$$(3.9) \qquad \frac{f(x, y_x)}{g(x, y_x)} \leq t',\ \forall x \in X.$$

Therefore, from (3.8) and (3.9), we obtain:

$$h(x', y') = \min_{y \in Y} h(x', y),$$

and

$$h(x', y') \geq h(x, y_x) \geq \min_{y \in Y} h(x, y),\ \forall x \in X,$$

what means, by Definition 2.1, that (x', y') is an optimal solution for the FD problem.

The necessity part of this theorem can be proved in a similar manner.

THEOREM 3.2. *If the assumption H1 holds, then:*

(*i*) $F(t) = 0$ *if and only if* $v = t$;

(*ii*) $F(t) > 0$ *if and only if* $v > t$;

(*iii*) $F(t) < 0$ *if and only if* $v < t$.

Proof. The statement (*i*) results by Theorem 1, [12]. We prove now the statement (*ii*). Thus, by (3.1), the inequality $F(t) > 0$ is equivalent to the existence of an element $x' \in X$, such that:

$$\min_{y \in Y} (f(x', y) - tg(x', y)) > 0,$$

which, in its turn, is equivalent to:

$$(3.10) \qquad f(x', y) - t \cdot g(x', y) > 0,\ \forall y \in Y.$$

But by $H1$, the inequality (3.10) is equivalent to:

$$\frac{f(x', y)}{g(x', y)} > t,\ \forall y \in Y,$$

or

$$(3.11) \qquad \min_{y \in Y} \frac{f(x', y)}{g(x', y)} \geq t.$$

From Definition 2.1 and (3.11), holds:

(3.12) $$v \geq \min_{y \in Y} h(x', y) \geq t.$$

Since, by part (*i*), $v = t$ is equivalent to $F(t) = 0$, it results, from (3.12) that $F(t) > 0$ if and only if $v > t$.

The part (*iii*) of the theorem is an obvious consequence of the parts (*i*) and (*ii*).

4. Parametrical procedure

The algorithm below is similar to that used in the case of the usual nonlinear fractional programming [3], [7], [8] or of nonlinear piecewise fractional programming [9].

Algorithm 1

Initial phase:

Step 1. Choose $x_0 \in X$ and take $k := 0$.

Step 2. Find $t_0 \in \mathbf{R}$ and $y_0 \in Y$ such that:

(4.1) $$t_0 = h(x_0, y_0) = \min_{y \in Y} h(x_0, y).$$

General phase:

Step 3. Find the optimal solution (x'_{k+1}, y'_{k+1}) of the following max-min problem:

(4.2) $$F(t_k) = \max_{x \in X} \min_{y \in Y} (f(x, y) - t_k\, g(x, y)).$$

Step 4. *i*) If $F(t_k) = 0$, then stop. By Theorem 3.2, (x_k, y_k) is an optimal solution for the *FD* problem.

ii) If $F(t_k) > 0$, then take $x_{k+1} = x'_{k+1}$ and go to Step 5.

Step 5. Find $t_{k+1} \in \mathbf{R}$ and $y_{k+1} \in Y$, such that:

(4.3) $$t_{k+1} = h(x_{k+1}, y_{k+1}) = \min_{y \in Y} h(x_{k+1}, y).$$

Step 6. Take $k := k + 1$ and go to Step 3.

5. Convergence of the algorithm 1

In this section some sufficient conditions for the convergence of the parametrical procedure presented in the preceding section are derived.

THEOREM 5.1. ([12]) *If the assumption H*1 *holds, then:*

(5.1) $$t_{k+1} - t_k \geq \frac{F(t_k)}{g(x_{k+1}, y_{k+1})}.$$

THEOREM 5.2. *If the assumption H*1 *holds then the algorithm* 1 *ends after a finite number of iterations.*

Proof. By Theorem 5.1, for every natural k, such that $F(t_k) > 0$, we have, from (5.1), the inequality: $t_{k+1} > t_k$.

Then from (4.1) and (4.3), it follows that the algorithm 1 generates a sequence of points:

(5.2) $$(x_0, y_0), (x_1, y_1), \ldots, (x_k, y_k), \ldots$$

having the property:

(5.3) $$h(x_0, y_0) < h(x_1, y_1) < \ldots < h(x_k, y_k) < \ldots.$$

Since the set $X \times Y$ is a finite set, then by the inequalities (5.3) it results that the sequence (5.2) is finite. Therefore the algorithm 1 ends after a finite number of iterations.

6. Fractional max-min problems in graphs

Let $G = (X, W)$ be a finite graph, where X denotes the vertices set and W the arcs set. Let $\mathfrak{D}$ and $\mathfrak{E}$ be two nonvoid sets containing some subgraphs of the graph G. For instance, $\mathfrak{D}$ and $\mathfrak{E}$ can be the sets of all the paths between the vertices $p, q (p \in X, q \in X)$ and $p'_1, q'_1 (p'_1 \in X, q'_1 \in X)$ respectively.

On the set W there are given the functions $p' : W \to \mathbf{R}$, $p'' : W \to \mathbf{R}^+$, $q' : W \to \mathbf{R}$, $q'' : W \to \mathbf{R}^+$, i.e. each arc of the graph G is weighted with four real weights.

In the following to simplify the notation, if w is an arc of a subgraph H of G, we will write $w \in H$.

We consider the following fractional max-min problem on the graph G: *PG*. Find

$$v = \max_{D \in \mathfrak{D}} \min_{E \in \mathfrak{E}} \frac{\sum_{w \in D} p'(w) + \sum_{w \in E} q'(w)}{\sum_{w \in D} p''(w) + \sum_{w \in E} q''(w)}$$

For solving the *PG* problem a variant of the algorithm 1 will be presented.

Algorithm 2

Initial phase:

Step 1. Choose $D_0 \in \mathfrak{D}$ and take $k := 0$.

Step 2. Find $t_0 \in \mathbf{R}$, such that:

(6.1) $$t^0 = \min_{E \in \mathfrak{E}} \frac{a'_0 + \sum_{w \in E} q'(w)}{a''_0 + \sum_{w \in E} q''(w)}$$

where:

$$a'_k = \sum_{w \in D_k} p'(w),\ a''_k = \sum_{w \in D_k} p''(w), \tag{6.2}$$

for every natural number k.

Let E_0 be an optimal solution of the problem (6.1).

General phase:

Step 3. Find

$$r = \max_{D \in \mathfrak{D}} \sum_{w \in D} (p'(w) - t_k \cdot p''(w)). \tag{6.3}$$

Let $D_{k+1} \in \mathfrak{D}$ be an optimal solution of the problem (6.3).

Step 4. Find

$$t_{k+1} = \min_{E \in \mathfrak{E}} \frac{a'_{k+1} + \sum_{w \in E} q'(w)}{a''_{k+1} + \sum_{w \in E} q''(w)}, \tag{6.4}$$

where a'_{k+1}, a''_{k+1} are defined by (6.2).

Step 5. *i*) If $t_{k+1} - t_k = 0$, then stop. The pair (D_k, E_k) is an optimal solution of the *PG* problem.

ii) If $t_{k+1} - t_k > 0$, then go to Step 6.

Step 6. Take $k := k + 1$ and go to Step 3.

Remarks

1. The algorithm 2 has some improvement in respect to the algorithm 1. Thus, at the step 3 of the algorithm 2, only a maximum subgraph problem (see, the problem (6.3)) must be solved, while in the algorithm 1, it must be solved a max-min problem (see, the problem (4.2)).

Also, the decision steps (i.e. Step 4 in the algorithm 1 and Step 5 in the algorithm 2) are differently formulated in these algorithms, but they are equivalent by the theorems 3.2 and 5.1.

2. The problem (6.4) (or (6.1)) at the step 4 (or the step 2) of the algorithm 2 is a fractional minimum subgraph problem, for which, in some particular cases, such as when $\mathfrak{E}$ is the set of all paths between two given vertices of a graph G without circuits (or the set of all spanning trees) there exists efficient algorithms (see, [1], [4], [5], [6], [11]).

REFERENCES

[1] Chandrasekaran, R., *Minimal Ratio Spanning Trees*, Networks **7**, 335–342 (1977).

[2] Cook, W. D., Kirby, M. J. L., Mehndiratta, S. L., *A linear Fractional Max-min Problem*, Operations Res., **23**, 3, 511–521 (1975).

[3] Dinkelbach, W., *On nonlinear Fractional Programming*, Manag. Sci., **13**, 7, 492–498 (1967).

[4] Gondran, M., Minoux, M., *Graphes et algorithmes*, Ed. Eyrolles, Paris (1979).

[5] Gondran, M., *Le problèmes de ratio minimum en optimisation combinatoire*, Preprint E.D.F., 1980.

[6] Peteanu, V., Ţigan, S., *O nouă problemă de drum optim în grafe*, Al II-lea Simpozion ,,Informatică şi Conducere'', 1976, Cluj-Napoca, vol. II, 169–173 (1977).

[7] Schaible, S., *Duality in Fractional Programming. A Unified Approach*, Operations Res., **24**, 3, 452–461 (1976).

[8] Shepilov, M. L., *On Methods for solving Fractional Problems of Mathematical Programming*, Kibernetika, **1**, 93–98 (1980).

[9] Ţigan, S., *Sur une méthode pour la résolution d'un problème d'optimisation fractionnaire par segments*, Mathematica-Revue d'analyse numérique et de théorie de l'approximation, **4**, **1**, 87–97 (1975).

[10] Ţigan, S., *On the max-min nonlinear fractional problem*, Mathematica – Revue d'analyse numérique et de théorie de l'approximation, **9**, 2, 283–288 (1980).

[11] Ţigan, S., *Problema fracţionară a arborelui minim*, Bul. Rom. Inf., **1**, 9–15 (1982).

[12] Ţigan, S., *A parametrical method for max-min nonlinear fractional problems*, Itinerant seminar on functional equations, approximation and convexity, Cluj-Napoca, 1983, Preprint no. 2, Babeş-Bolyai Univ., 175–184 (1983).

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