MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 13, N° 2, 1984, pp. 175—187

THE MINIMUM RISK APPROACH TO SPECIAL PROBLEMS OF MATHEMATICAL PROGRAMMING. THE DISTRIBUTION FUNCTION OF THE OPTIMAL VALUE

by

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1. Introduction

In this paper some special classes of mathematical programming problems as Chebyshev problems, linear plus indefinite (or fractional) programming problems are considered. For such problems the distribution function of the optimal value is determined and the minimum risk approach to them is presented. Firstly, a general model of nonlinear minimum risk problem will be discussed, followed by the treatment of the stated problems.

2. General formulation of nonlinear minimum risk problem

The general problem described here can be mathematically expressed as:

Problem P:

(2.1) Minimize $f(a(t), x)$ subject to $x \in S$,
where the set $S \subseteq \mathbf{R}^n$, the real vector valued function $a: A \to B$, the real valued function $f: B \times S \to \mathbf{R}$, the sets $A \subseteq \mathbf{R}$ and $B \subseteq \mathbf{R}^m$ are assumed to be deterministic and known (in particular B may be the whole space $\mathbf{R}^n$, i.e. $B \equiv \mathbf{R}^n$).

We shall make the following assumption concerning this problem.
(A.1) There exists a function $g: S \times C \to C$, $C \subseteq \mathbf{R}$ such that for all $x \in S$

and for all $t \in A$

$$f(a(t),\ x) \leqslant w \Leftrightarrow t \leqslant g(x,\ w)$$

where w is an arbitrary real number in C.

In fact, the above condition assures the separation of t from inequality $f(a(t),\ x) \leqslant w$.

Now, we consider that $t(\omega)$ is a random variable on a given probability space $(\Omega,\ K,\ P)$ with a continuous and strictly increasing distribution function $T(z)$.

Throughout this paper we consider stochastic programs with simple randomization (see [1], [2]) that is programs, the random coefficients of which are affine functions of a simple random variable.

The minimum risk approach to the problem (P) is to find the solution of the following programming problem:

Problem PRM:

$$\max_{x \in S} P\{\omega / f(a(t(\omega)),\ x) \leqslant w\}. \tag{2.2}$$

Then, as in STANCU–MINASIAN [6], we can establish a relationship between the minimum risk problem associated with the Problem P corresponding to the level w (Problem PRM) and a deterministic problem which does not depend on distribution function of the random variable $t(\omega)$.

From our assumption (A.) it follows that

$$P\{w / f(a(t(\omega)),\ x) \leqslant w\} = P\{\omega / t(\omega) \leqslant g(x,\ w)\} = T(g(x,\ w)).$$

Further, according to the fact T is an increasing function, we have:

$$\max_{x \in S} P\{\omega / f(a(t(\omega)),\ x) \leqslant w\} = \max_{x \in S} T(g(x,\ w)) = T(\max_{x \in S} g(x,\ w)).$$

We have the following theorem.

THEOREM 1. *If the distribution function $T(z)$ of $t(\omega)$ is continuous strictly increasing then the minimum risk solution coresponding to the level w of the* Problem PRM, *does not depend on $T(z)$ and can be determined by solving the deterministic problem:*

Problem PA:

$$\max_{x \in S} g(x,\ w) \tag{2.3}$$

The subsequent sections of this paper will illustrate this technique for some special classes of problems of mathematical programming as Chebyshev problems, linear plus indefinite (or fractional) programming problems. The distribution function of the optimal value i.e. of the random variable

$$\xi(\omega) = \min_{x \in S} f(a(t(\omega)),\ x)$$

is also presented. So, it is necessary to find out characteristic values (λ_j) and characteristic solutions (x^j) of the parametric program, when in Problem P, $t(\omega)$ is replaced by the parameter λ.

3. Stochastic linear-plus-fractional programming problem

Consider the following problem:

$$\max\left(c'x + \frac{e'x}{d'x}\right) \tag{3.1}$$

subject to

$$x \in S = \{x / Ax = b,\ x \geqslant 0\}, \tag{3.2}$$

where A is an $(m \times n)$ matrix $(m < n)$ of rank m, c, d, e, x, are $(n \times 1)$ vectors, b is $(m \times 1)$ vector and prime (') denotes transposition. It arises when the remuneration fund and the profitableness of an economic entreprise should be optimized.

Under the assumptions $c'x \geqslant 0$ and $e'x > 0$ for all $x \in S$, a simplex type algorithm has been proposed by TETEREV [9].

Let

$$c(\omega) = c_1 + t(\omega)c_2$$
$$d(\omega) = d_1 + t(\omega)d_2$$
$$e(\omega) = e_1 + t(\omega)e_2$$

where c_i, d_i, $e_i (i = 1,\ 2)$ are constant $(n \times 1)$ vectors and $t(\omega)$ is a random variable.

For the stochastic linear-plus-fractional programming problem we make the following assumptions:

(A3) The set of feasible solutions S is regular i.e. nonvoid and bounded.
(B3) The denominator of the objective function preserves the same sign (let us assume it to be positive) on S, i.e.

$$P\{\omega / (e_1 + t(\omega)e_2)'x > 0\} = 1.$$

Also, we assume that

$$P\{\omega / (c_1 + t(\omega)c_2)'x \geqslant 0\} = 1.$$

(C3) Every basic feasible solution is non-degenerate.

We shall consider the following types of stochastical problems associated to the problem (3.1)–(3.2):

i) One of the vectors c, d or e is random.
ii) The vectors c and d or c and e or d and e are random.
iii) The vectors c, d and e are random.

Because *i*) is a special case of *ii*) we shall study the last one.

Case ii). The vectors c and d are random.

First, let us determine the distribution function of the optimal value i.e. of the random variable

$$\xi(\omega) = \max_{x \in S} \left[(c_1 + t(\omega)\, c_2)' x + \frac{(d_1 + t(\omega)\, d_2)' x}{e' x} \right] \tag{3.3}$$

We assume $t(\omega) = \lambda$ is a parameter varying within an interval $[\delta_1, \delta_2]$.

It is well known that for parametric linear-plus-fractional problem of the from (3.3) the interval $[\delta_1, \delta_2]$ may be divided into a finite number of so-called critical regions characterized by the various combinations of variables forming optimal bases (see [3], [4]). Let $\lambda_j (1 \leqslant j \leqslant p-1)$ be the characteristic values

$$\delta_1 \leqslant \lambda_1 < \lambda_2 < \ldots < \lambda_{p-1} \leqslant \delta_2$$

and x^j $(1 \leqslant j \leqslant p)$ the characteristic solutions. We denote $\delta_1 = \lambda_0$ and $\delta_2 = \lambda_p$. Moreover, due to assumption that the random variable $t(\omega)$ has a continuous distribution function, the intersection of two such critical regions has zero probability.

We have the following theorem:

THEOREM 2. *Let $F(z)$ ge the distribution function of $\xi(\omega)$ and let*

$$u_j(z) = \frac{z(e' x^j) - d_1' x^j - (c_1' x^j)(e' x^j)}{d_2' x^j + (c_2' x^j)(e' x^j)} .$$

Then

$$F(z) = \sum_{j=1}^{p} H_j(z),$$

where

$$H_j(z) = \begin{cases} T(\lambda_j) - T(\lambda_{j-1}) & \text{if } \begin{cases} d_2'x^j + (c_2'x^j)(e'x^j) > 0 \text{ and } u_j(z) \geqslant \lambda_j \\ d_2'x^j + (c_2'x^j)(e'x^j) < 0 \text{ and } u_j(z) \leqslant \lambda_{j-1} \\ d_2'x^j + (c_2'x^j)(e'x^j) = 0 \text{ and } \\ \qquad d_1'x^j + (c_1'x^j)(e'x^j) < z(e'x^j) \end{cases} \\ 0 & \text{if } \begin{cases} d_2'x^j + (c_2'x^j)(e'x^j) > 0 \text{ and } u_j(z) \leqslant \lambda_{j-1} \\ d_2'x^j + (c_2'x^j)(e'x^j) < 0 \text{ and } u_j(z) \geqslant \lambda_j \\ d_2'x^j + (c_2'x^j)(e'x^j) = 0 \text{ and } \\ \qquad d_1'x^j + (c_1'x^j)(e'x^j) \geqslant z(e'x^j) \end{cases} \\ \left.\begin{matrix} T(u_j(z)) - T(\lambda_{j-1}) \\ T(\lambda_j) - T(u_j(z)) \end{matrix}\right\} & \text{if } \lambda_{j-1} < u_j(z) < \lambda_j \text{ and } \begin{cases} d_2'x^j + (c_2'x^j)(e'x^j) > 0 \\ d_2'x^j + (c_2'x^j)(e'x^j) < 0 \end{cases} \end{cases} \quad (1 \leqslant j \leqslant p)$$

Proof. Consider the events $A = \{\omega / \xi(\omega) < z\}$ and $B_j = \{\omega / \lambda_{j-1} \leqslant t(\omega) < \lambda_j\}$. Because $\bigcup_{i=1}^{p} B_i = \Omega$ and $P(B_j) > 0$ $(1 \leqslant j \leqslant p)$ $(B_i \cap B_j \neq 0, i \neq j)$ one can use the formula of total probability:

$$P(A) = \sum_{j=1}^{p} P(A|B_j)\, P(B_j)$$

that is

$$F(z) = P(A) = \sum_{j=1}^{p} P(A \cap B_j) = \sum_{j=1}^{p} H_j(z)$$

where

$$H_j(z) = P\left\{\omega / \left[\omega | (c_1 + t(\omega) c_2)' x^j + \frac{(d_1 + t(\omega)\, d_2)' x^j}{e' x^j} < z \right] \cap \right.$$

$$\left. \cap\ [\omega | \lambda_{j-1} \leqslant t(\omega) < \lambda_j] \right\}$$

a) If $d_2'x^j + (c_2'x^j)(e'x^j) > 0$ then

$$(c_1' + t(\omega)\, c_2')' x^j + \frac{(d_1 + \omega t(\omega)\, d_2)' x^j}{e' x^j} < z$$

when $t(\omega) < \dfrac{z(e' x^j) - d_1' x^j - (c_1' x^j)(e' x^j)}{d_2' x^j + (c_2' x^j)(e' x^j)}$ so that

$$H_j(z) = 0 \text{ if } u_j(z) \leqslant \lambda_{j-1}$$

$$H_j(z) = T(\lambda_j) - T(\lambda_{j-1}) \text{ if } u_j(z) \geqslant \lambda_j$$

and

$$H_j(z) = T(u_j(z)) - T(\lambda_{j-1}) \text{ if } \lambda_{j-1} < u_j(z) < \lambda_j.$$

b) If $d_2'x^j + (c_2'x^j)(e'x^j) < 0$ then

$$(c_1 + t(\omega)\, c_2)' x^j + \frac{(d_1 + t(\omega)\, d_2)' x^j}{e' x^j} < z$$

when $t(\omega) > \dfrac{z(e' x^j) - d_1' x^j - (c_1' x^j)(e' x^j)}{d_2' x^j + (c_2' x^j)(e' x^j)}$ so that

$$H_j(z) = 0 \text{ if } u_j(z) \leqslant \lambda_j$$

$$H_j(z) = T(\lambda_j) - T(\lambda_{j-1}) \text{ if } u_j(z) \leqslant \lambda_{j-1}$$

and

$$H_j(z) = T(\lambda_j) - T(u_j(z)) \text{ if } \lambda_{j-1} < u_j(z) < \lambda_j.$$

c) If $d_2'x^j + (c_2'x^j)(e'x^j) = 0$ then

$$H_j(z) = 0 \text{ if } d_1'x^j + (c_1'x^j)(e'x^j) \geq z(e'x^j)$$

and

$$H_j(z) = T(\lambda_j) - T(\lambda_{j-1}) \text{ if } d_1'x^j + (c_1'x^j)(e'x^j) < z(e'x^j).$$

Summarizing, we get the theorem.

In the remainder of the paragraph we shall consider the minimum risk approach to the problem (3.1) — (3.2) wich consists in finding the optimal solution of the following programming problem:

$$\text{(3.4)} \qquad v(w) = \max_{x \in S} P\left\{\omega/(c_1 + t(\omega)c_2)'x + \frac{(d_1 + t(\omega)x_2)'x}{e'x} \geq w\right\}.$$

We shall make a further assumption

$$\text{(D.3)} \qquad h(x) = (c_2'x)(e'x) + d_2'x \neq 0 \text{ for all } x \in S.$$

Let

$$g(x, w) = \frac{(we - d_1)'x - (c_1'x)(e'x)}{(c_2'x)(e'x) + d_2'x}.$$

Obviously it is true that

$$F(x, w) = P\left\{\omega/(c_1 + t(\omega)c_2)'x + \frac{(d_1 + t(\omega)d_2)'x}{e'x} \geq w\right\} =$$

$$= \begin{cases} P\{\omega/t(\omega) \leq g(x, w)\} \text{ if } h(x) < 0 \\ P\{\omega/t(\omega) \geq g(x, w)\} \text{ if } h(x) > 0 \end{cases} = \begin{cases} T(g(x, w)) \text{ if } h(x) < 0 \\ 1 - T(g(x, w) \text{ if } h(x) > 0 \end{cases}$$

and therefore, using the assumption that $T(z)$ is strictly increasing:

$$\max_{x \in S} F(x, w) = \begin{cases} T(\max_{x \in S} g(x, w)) \text{ if } h(x) < 0 \\ 1 - T(\min_{x \in S} g(x, w)) \text{ if } h(x) < 0. \end{cases}$$

We then have:

THEOREM 3. *If the assumptions* (A3)—(D3) *are true and the distribution function* $T(z)$ *of* $t(\omega)$ *is continuous strictly increasing, then the minimum risk solution of problem* (3.4) *corresponding to the level* w *does not depend on* $T(z)$ *and can be determined by solving the deterministic problem*

$$\text{(3.5)} \qquad \max_{x \in S} g(x, w) \quad \text{if} \quad h(x) < 0$$

or

$$\text{(3.6)} \qquad \min_{x \in S} g(x, w) \quad \text{if} \quad h(x) > 0.$$

Thus the minimum risk approach to linear-plus-fractional programming leads to a nonlinear fractional programming problem.

Remarks: *a*) Taking $d_2 = 0$ or $c_2 = 0$ we obtain the case *i*).

b) If the vectors d and e are random, the minimum risk problem corresponding to the problem (3.1)—(3.2) has the form

$$\text{(3.7)} \qquad \max_{x \in S} P\left\{\omega/c'x + \frac{(d_1 + t(\omega)d_2)'x}{(e_1 + r(\omega)e_2)'x} \geq w\right\}.$$

Let

$$q(x, w) = \frac{(we_1 - d_1)'x - (c'x)(e_2 x)}{(d_2 - we_2)'x + (c'x)(e_2'x)}, \text{ for all } x \in S.$$

If the assumptions (A3)—(C3) and

$$\text{(E3)} \qquad h_1(x) = (d_2 - we_2)'x + (c'x)(e_2'x) \neq 0 \text{ for all } x \in S,$$

are true, the problem (3.7) is equivalent to

$$\text{(3.8)} \qquad \max_{x \in S} q(x, w) \quad \text{if} \quad h_1(x) < 0$$

or

$$\text{(3.9)} \qquad \min_{x \in S} q(x, w) \quad \text{if} \quad h_1(x) > 0.$$

For $c = 0$ we obtain the minimum risk problem corresponding to linear fractional programming problem. In [6], [7] is studied the problem of the distribution function of the random variable

$$\xi(\omega) = \max_{x \in S} (\min) \left[\frac{(d_1 + t(\omega)d_2)'x}{(e_1 + t(\omega)e_2)'x}\right].$$

Case iii). The minimum risk approach to the problem (3.1), (3.2) is the following problem

$$\text{(3.10)} \qquad v(w) = \max_{x \in S} P\left\{\omega/(c_1 + t(\omega)c_2)'x + \frac{(d_1 + t(\omega)d_2)'x}{(e_1 + t(\omega)e_2)'x} \geq w\right\}.$$

In this case it can not be found a deterministic problem independent of the distribution function of the random variable $t(\omega)$ whose solution could be a solution of the minimum risk problem (3.10). Therefore, we shall find the distribution function of the optimal value i.e. of the random variable

$$\xi(\omega) = \max_{x \in S} \left[(c_1 + t(\omega)c_2)'x + \frac{(d_1 + t(\omega)d_2)'x}{(e_1 + t(\omega)e_2)'x}\right].$$

As in the case *ii*) we assume $t(\omega) = \lambda$ and let λ_j $(1 \leq j \leq p)$ and $x^j (1 \leq j \leq p)$ be the characteristic values and solutions respectively of the parametric program.

From

$$(c_1 + \lambda c_2)'x^j + \frac{(d_1 + \lambda d_2)'x^j}{(e_1 + \lambda e_2)'x^j} = z$$

and (B3) it results

$$u_j\lambda^2 + v_j(z)\lambda + w_j(z) = 0,$$

where

$$u_j = (c_2' x^j)(e_2' x^j)$$
$$v_j(z) = (c_2' x^j)(e_1' x^j) + (c_1' x^j)(e_2' x^j) - d_2' x^j - z(e_2' x^j)$$
$$w_j(z) = (c_1' x^j)(e_1' x^j) - z \cdot (e_1' x^j) - d_1' x^j.$$

Let $\lambda_j^1(z)$, $\lambda_j^2(z)$ be the roots of this equation. We assume, when these roots are real, that

$$\lambda_j^1(z) \leqslant \lambda_j^2(z).$$

We have the following theorem:

THEOREM 4. *Let $F(z)$ be the distribution function of $\xi(\omega)$ and $u_j''(z) = -\frac{w_j(z)}{v_j(z)}$. Then $F(z) = \sum_{j=1}^{p} H_j(z)$, where we have*

$$t_j^i = \min(\lambda_j, \lambda_j^i(z)),\ r_j^i = \max(\lambda_{j-1}, \lambda_j^i(z)),\ (1 \leqslant j \leqslant p),\quad i = 1,2$$

and

$$H_j(z) = \begin{cases} T(t_j^2) - T(r_j^1) \text{ when } u_j^1 > 0 \text{ and } r_j^2 < t_j^2 \\ T(\lambda_j) - T(r_j^2) + T(t_j^1) - T(\lambda_{j-1}) \text{ when } u_j < 0,\ r_j^2 < \lambda_j,\ \lambda_{j-1} < t_j^1 \\ \left.\begin{matrix} T(\min(u_j''(z), \lambda_j)) - T(\lambda_{j-1}) \\ T(\lambda_j) - T(\max(u_j''(z), \lambda_{j-1}))\end{matrix}\right\} \text{ when } u_j = 0 \begin{cases} V_j(z) > 0,\ \lambda_{j-1} < u_j''(z) \\ v_j(z) < 0,\ u_j''(z) < \lambda_j \end{cases} \\ T(t_j^1) - T(\lambda_{j-1}) \text{ when } u_j < 0 \text{ and } t_j^1 \leqslant \lambda_{j-1} \\ T(\lambda_j) - T(\lambda_{j-1}) \text{ when } \begin{cases} \lambda_j^i(z) \text{ are complex or equal and } u_j < 0 \\ u_j = 0,\ v_j(z) = 0 \text{ and } w_j(z) < 0 \end{cases} \\ 0 \text{ when } \begin{cases} u_j > 0 \text{ and } \begin{cases} \lambda_j^i(z)\ (i = 1, 2) \text{ are complex or equal} \\ r_j^1 \geqslant t_j^2 \end{cases} \\ u_j < 0 \text{ and } \lambda_j \leqslant r_j^2,\ t_j^1 < \lambda_{j-1} \\ u_j = 0 \text{ and } \begin{cases} v_j(z) > 0 \text{ and } \lambda_{j-1} \geqslant u_j''(z) \\ v_j(z) < 0 \text{ and } u_j''(z) \geqslant \lambda_j \\ v_j(z) = 0 \text{ and } w_j(z) \geqslant 0. \end{cases} \end{cases} \end{cases}$$

$(1 \leqslant j \leqslant p)$

Proof. The proof is similar to that of Theorem 2. Here we have

$$H_j(z) = P\{[\omega \mid u_j\lambda^2 + v_j(z)\lambda + w_j(z) < 0] \cap$$
$$\cap\ [\omega \mid \lambda_{j-1} \leqslant \lambda \leqslant \lambda_j]\}(1 \leqslant j \leqslant p).$$

Let

$$S_1 = \{\lambda \in [\delta_1, \delta_2] \mid u_j\lambda^2 + v_j(z)\lambda + w_j(z) < 0 \text{ and } \lambda_{j-1} \leqslant \lambda < \lambda_j\}.$$

It is easy to see that

$$S_1 = \begin{cases} \begin{Bmatrix} \varnothing \\ [\lambda_{j-1}, \lambda_j) \end{Bmatrix} \text{ if } \lambda_j^i(z)(i = 1, 2) \text{ are complex or equal and } \begin{cases} u_j > 0 \\ u_j < 0 \end{cases} \\ \begin{Bmatrix} [r_j^1, t_j^2] \\ [r_j^2, \lambda_j] \cup [\lambda_{j-1}, t_j^1] \end{Bmatrix} \text{ if } \begin{Bmatrix} u_j > 0 \\ u_j < 0 \end{Bmatrix} \\ \begin{Bmatrix} \left[\lambda_{j-1}, \min\left(\frac{-w_j(z)}{v_j(z)}, \lambda_j\right)\right] \\ \left[\max\left(\frac{-w_j(z)}{v_j(z)}, \lambda_{j-1}\right), \lambda_j\right] \end{Bmatrix} \text{ if } u_j = 0 \text{ and } \begin{cases} v_j(z) > 0 \\ v_j(z) < 0 \end{cases} \\ \begin{Bmatrix} [\lambda_{j-1}, \lambda_j] \\ \varnothing \end{Bmatrix} \text{ if } u_j = v_j(z) = 0 \text{ and } \begin{cases} 0 > w_j(z) \\ 0 \leqslant w_j(z). \end{cases} \end{cases}$$

Taking into account these evaluations in $H_j(z)$ the theorem results immediately.

4. Stochastic Chebyshev problem

Given the functions

$$z_i(x) = c^{i\prime} x + \alpha^i \quad (i = 1, 2, \ldots, r)$$

where $c^i (i = 1, 2, \ldots, r)$ are constant vectors and $\alpha^i (i = 1, 2, \ldots, r)$ are constants, we consider the function

$$z(x) = \max_{1 \leqslant i \leqslant r} \{z_i(x)\}.$$

By the Chebyshev problem we usually understand a problem of the type:

$$\min_x \max_{1 \leqslant i \leqslant r} \{z_i(x)\} \tag{4.1}$$

subject to

$$x \in S = \{x / Ax = g,\ x \geqslant 0\}.$$

The notations have the same meaning as in previous paragraphs. Let us consider:

$$c^i(\omega) = c_1^i + t(\omega)\, c_2^i$$
$$\alpha^i(\omega) = \alpha_1^i + t(\omega)\, \alpha_2^i$$

where c_1^i, $c_2^i (i = 1, 2, \ldots, r)$ are constant vectors, α_1^i, α_2^i $(i = 1, 2, \ldots, r)$ are scalar constants and $t(\omega)$ is a random variable.

We make the following assumption:

(A4) $c_2^i x + \alpha_2^i \neq 0$, for all $x \in S$ and $i \in M = \{1, 2, \ldots, r\}$.

The minimum risk approach to the Chebyshev problem (4.1), (4.2) consists in finding the optimal solution of the following programming problem:

$$(4.3) \qquad v(w) = \max_{x \in S} P\{\omega / \max_{x \in M} [(c^i(\omega))' x + \alpha^i(\omega)] \leqslant w\}.$$

Let $z_i'(x) = c_2^{i'} x + \alpha_2^i$,

$$g_i(x, w) = \frac{w - c_1^{i'} x - \alpha_1^i}{c_2^{i'} x + \alpha_2^i}, \text{ for all } x \in S \text{ and } i \in M.$$

Then we have:

$$F(x, w) = P\{\omega / \max_{i \in M} [(c_1^i + t(\omega) c_2^i)' x + \alpha_1^i + t(\omega)\alpha_2^i] \leqslant w\} =$$

$$= P\{\omega / (c_1^i + t(\omega) c_2^i)' x + \alpha_1^i + t(\omega)\alpha_2^i \leqslant w, \ \forall i \in M\} =$$

$$= \begin{cases} P\{\omega / t(\omega) \leqslant g_i(x, w), \ \forall i \in M\} \text{ if } z_i'(x) > 0 \\ P\{\omega / t(\omega) \geqslant g_i(x, w), \ \forall i \in M\} \text{ if } z_i'(x) < 0 \end{cases} =$$

$$= \begin{cases} P\{\omega / t(\omega) \leqslant \min_{i \in M} g_i(x, w)\} \text{ if } z_i'(x) > 0 \\ P\{\omega / t(\omega) \geqslant \max_{i \in M} g_i(x, w)\} \text{ if } z_i'(x) < 0 \end{cases} =$$

$$= \begin{cases} T(\min_{i \in M} g_i(x, w)) \text{ if } z_i'(x) > 0 \\ 1 - T(\max_{i \in M} g_i(x, w)) \text{ if } z_i'(x) < 0. \end{cases}$$

Hence our problem will be

$$\max_{x \in S} F(x, w) = \begin{cases} \max_{x \in S} T(\min_{i \in M} g_i(x, w)) \text{ if } z_i'(x) > 0 \\ 1 - \min_{x \in S} T(\max_{i \in M} g_i(x, w)) \text{ if } z_i'(x) < 0, \end{cases}$$

But if the distribution function $T(z)$ of the random variable $t(\omega)$ is continuous and strictly increasing, we have:

$$v(w) = \max_{x \in S} F(x, w) = \begin{cases} T(\max_{x \in S} \min_{i \in M} g_i(x, w)), \text{ if } z_i'(x) > 0 \\ 1 - T(\min_{x \in S} \max_{i \in M} g_i(x, w)), \text{ if } z_i'(x) < 0. \end{cases}$$

Thus, one gets immediately the following theorem:

THEOREM 5. *If the assumption* (A4) *occurs and the distribution function* $T(z)$ *of* $t(\omega)$ *is continuous strictly increasing, then the minimum risk solution of problem* (4.3) *does not depend on* $T(z)$ *and can be determined by solving the deterministic piecewise linear fractional programming problem:*

$$(4.4) \qquad \max_{x \in S} \min_{i \in M} \frac{w - c_1^{i'} x - \alpha_1^i}{c_2^{i'} x + \alpha_2^i} \quad \textit{if } z_i'(x) > 0$$

or

$$(4.5) \qquad \min_{x \in S} \max_{i \in M} \frac{w - c_1^{i'} x - \alpha_1^i}{c_2^{i'} x + \alpha_2^i} \quad \textit{if } z_i'(x) < 0.$$

Remarks 1) The piecewice linear fractional programming problems (4.4) or (4.5) can be solved by a parametrical algorithm (see [10]) similar to the Dinkelbach's algorithm for fractional programming (see [5]).

2) In [8] STANCU – MINASIAN determines the distribution function of the optimal value for stochastic Chebyshev problem considering the same hypothesis that the random coefficients are affine functions of a single random variable.

3) When

$$c^i(\omega) = c_1^i + t^i(\omega) c_2^i, \ i \in M, \ \alpha^i = 0, \ (i \in M),$$

and $t^i(\omega)$ $(i \in M)$ are independent random variables with the distribution functions T_i continuous strictly increasing then the minimum risk solution of problem (4.3) depend on T_i as follows:

$$\max_{x \in S} F(x, w) = \max_{x \in S} \prod_{i=1}^{r} T_i\left(\frac{w - c_1^{i'} x}{c_2^{i'} x}\right).$$

4) When the functions z_i $(i \in M)$ defining the objective function z are nonlinear such as linear fractional, linear-plus-fractional or linear-plus-indefinite functions, the minimum risk approach to the Chebyshev problem can be restated and solved under appropriate hypothesis in a similar manner to the linear case.

5. Stochastic linear-plus-indefinite programming

Consider the following problem

$$(5.1) \qquad \min (c'x + d'x \cdot c'x)$$

subject to

$$(5.2) \qquad x \in S = \{x \in \mathbf{R}^n / Ax = b, \ x \geqslant 0\}$$

where the notations have the same meaning as in §3. Also, as in 3, we consider that the vectors c, d and e are affine functions of a single random variable $t(\omega)$, about which we make the same assumptions as in §2.

Then the minimum risk approach to the problem (5.1)—(5.2) consists in finding the optimal solution of the following problem:

$$\max_{x \in S} P\{\omega / c(\omega)'x + d(\omega)'x \cdot e(\omega)'x \leqslant w\}. \tag{5.3}$$

We shall consider the following cases: (*i*) c is random; (*ii*) the vectors c and d or c and e are random; (*iii*) the vectors d and e are random; (*iv*) the vectors c, d and e are random.

In order to simplify the exposition we make the following assumptions:

(A5) $\qquad c_2'x > 0, \text{ for all } x \in S;$

(B5) $\qquad c_2'x + d_2'x \cdot e'x > 0, \text{ for all } x \in S.$

Only in the cases (*i*) and (*ii*), the solution of the problem (5.3) does not depend on the distribution function of $t(\omega)$ and is got by solving a deterministic fractional programming problem. The cases (*iii*) and (*iv*) has not yet been solved.

Case (*i*). Using the assumption (A5) and the hypothesis that T is continuous strictly increasing, we have:

$$F(x, w) = P\{\omega / c(\omega)'x + d'x \cdot e'x \leqslant w\} =$$

$$= P\{\omega / c_1'x + t(\omega)\, c_2'x + d'x \cdot e'x \leqslant w\} =$$

$$= P\left\{\omega / t(\omega) \leqslant \frac{w - c_1'x - d'x \cdot e'x}{c_2'x}\right\} = T\left(\frac{w - c_1'x - d'x \cdot e'x}{c_2'x}\right).$$

Therefore

$$\max_{x \in S} F(x, w) = \max_{x \in S} T\left(\frac{w - c_1'x - d'x \cdot e'x}{c_2'x}\right) = T\left(\max_{x \in S} \frac{w - c_1'x - d'x \cdot e'x}{c_2'x}\right).$$

Hence, it follows the theorem:

THEOREM 6. *If* $c(\omega) = c_1 + t(\omega)c_2$, *and assumption* (A5) *holds and if the distribution function* $T(z)$ *of* $t(\omega)$ *is continuous strictly increasing, then the minimum risk solution of problem* (5.3) *does not depend on* $T(z)$ *and can be determined by solving the following problem:*

$$\max_{x \in S} \frac{c_1'x - d'x \cdot e'x}{c_2'x}. \tag{5.4}$$

Remark. In the special case when $c_2 = e$ the problem (5.4) can be reduced to a linear-plus-fractional programming problem.

Case (*ii*). The vectors c and d are random. The minimum risk problem associated with the problem (5.1)—(5.2) has the following form:

$$\max_{x \in S} P\{\omega / c_1'x + t(\omega) c_2'x + (d_1'x + t(\omega) d_2'x) e'x \leqslant w\}. \tag{5.5}$$

Following the same way as in the case (*i*) we have the theorem:

THEOREM 7. *Let* $c(\omega) = c_1 + t(\omega)c_2$ *and* $d(\omega) = d_1 + t(\omega)d_2$. *If the assumption* (B5) *holds and the distribution function* T *of* $t(\omega)$ *is continuous strictly increasing, then the minimum risk solution of problem* (5.5) *does not depend on* T *and can be determined by solving the following problem:*

$$\max_{x \in S} \frac{w - c_1'x - d_1' \cdot e'x}{c_2'x + d_2'x \cdot e'x}. \tag{5.6}$$

Remark. The problem of finding the distribution function of the optimal value i.e. of the random variable

$$\xi(\omega) = \min_{x \in S} [c(\omega)'x + d(\omega)'x \cdot e'(\omega)x]$$

can be done in any of the cases (*i*), (*ii*), similar to the previous paragraphs.

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Received 5.III.1983.

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