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ON SOME GENERALIZATIONS OF CONVEX SETS AND CONVEX FUNCTIONS

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1. INTRODUCTION

A set in a vector space is usually called convex if together with any two of its points it contains the whole interval joining them. At the same time, the applications of mathematics involve some extensions of this definition. Namely, in connection with the introduction of locally convex topologies J. von Neumann [11] requires only the midpoint of the interval to belong to this set. Then J.W Green and W. Gustin [6] and recently L.F. German and V.P. Soltan [5] claim that just the points dividing the interval in prescribed ratios remain in the set. I. Muntean [10] proves existence theorems of supporting hyperplanes to the sets which are convex in the latter sense. Finally, in establishing some fixed point theorems V.S. Shulman [13] introduces a concept of convexity stating that the intermediate points run a portion of a curve joining the end-points of the interval. We associate with these convexity notions for sets the corresponding convexity notions for real functions defined on such sets.

In this paper we study an attenuated convexity concept which includes the convexity notions introduced by J. von Neumann and J. W. Green and W. Gustin, and we put out some relations between this concept and the concept of usual convexity. A set Y in a vector space over the filed $\mathbb{R}$ or real numbers or the field $\mathbb{C}$ of complex numbers is said to be:

convex if for each $x, y \in Y$ and each p in $]0,1[$ we have

$$(1 - p)\,x + py \in Y; \tag{1.1}$$

p-convex, with p in $]0,1[$, if for each $x, y \in Y$ (1.1) holds;

weakly-convex if for each $x, y \in Y$ there exists a p in $]0,1[$ such that (1.1) holds.

Every convex set is p-convex for each p in $]0,1[$, and every p-convex set with a p in $]0,1[$ is weakly-convex. As the following examples show, there exist weakly-convex sets which are neither convex nor p-convex for any p in $]0,1[$.

Example 1.1. The set $\{0\} \cup]1,2]$ in $\mathbb{R}$ is weakly-convex without being p-convex for any p in $]0,1[$.

Example 1.2. Every open set Y in a topological vector space is weakly-convex. Indeed, if $x, y \in Y$, then the function $p \to (1-p)\,x + py,\ p \in [0,1]$, is continuous at the point $p = 0$ and Y is a neighborhood of x, hence there is a p in $]0,1[$ such that $(1-p)\,x + py \in Y$.

A real function f defined on a set Y in a vector space is said to be: convex if the set Y is convex and for each $x, y \in Y$ and each p in $]0,1[$ we have

$$f\left((1-px+py\right) \leq (1-p)\, f(x) + pf(y)\,; \tag{1.2}$$

p-convex, with p in $]0,1[$ if the set Y is p-convex and for each $x, y \in Y$ (1.2) holds;

weakly-convex if for each $x, y \in Y$ there exists a p in $]0,1[$ such that (1.1) and (1.2) hold.

In Section 2 of this paper we shall prove the convexity of weakly-convex closed sets, and the convexity of the closure of p-convex sets. Section 3 is devoted to the proof of an accessibility theorem for p-convex sets. In the last section we shall prove the convexity of the weakly-convex and lower-semicontinuous functions. By a counterexample we shall show that the theorems of A. Ostrowski [12], M.R. Mehdi [8] and E.Deák [3] concerning the convexity of p-convex functions which are bounded on sets of positive measure or on sets of second with Baire property fail for weakly-convex functions.

2. THE CONVEXITY OF WEAKLY-CONVEX SETS

THEOREM 2.1. *Every weakly-convex closed set in a topological vector space X is convex.*

Proof. Supposing the contrary, we can find a non-convex closed weakly-convex set Y in X. Then there are $x_0, y_0 \in Y$ and p_0 in $]0,1[$ such that $z(p_0) \notin Y$, where

$$z(p) = (1-p)\, x_0 + py_0.$$

Since the function $p \to z(p)$, $p \in [0,1]$, is continuous at $p = p_0$ and $Z = X \setminus Y$ is a neighbourhood of $z(p_0)$, there exists a $\delta > 0$ such that

$$z(p) \in Z \text{ for all } p \text{ in }]p_0 - \delta, p_0 + \delta[. \tag{2.1}$$

Denote $a_0 = \sup A$ where

$$A = \{a \in [0,1] : z(p) \in Z \text{ for all } p \text{ in } [p_0, a]\}.$$

From (2.1) we easily derive that $p_0 < a_0$. We shall show that $y = z(a_0) \in Y$. Suppose the contrary, i.e., $y \notin Y$. Then $a_0 < 1$ and we can find $\delta' > 0$ with $a_0 + \delta' < 1$ such that

$$z(p) \in Z \text{ for all } p \text{ in }]a_0 - \delta',\ a_0 + \delta'[\tag{2.2}$$

Let $a_1 \in A$ with $a_0 - \delta' < a_1 \leq a_0$. Then $a_2 = a_0 + \frac{\delta'}{2} \in A$ since $z(p) \in Z$ when $p \in [p_0, a_1]$, and $z(p) \in Z$ when $p \in]a_1, a_2[\subset]a_0 - \delta', a_0 + \delta'[$ by (2.2). Therefore, $a_2 \in A$ and we arrive at the contradiction $a_2 \leq a_0 < a_2$. Hence $y \in Y$.

Further denote $b_0 = \inf B$ where

$$B = \{b \geq 0 : z(p) \in Z \text{ for all } p \text{ in } [b, p_0]\}.$$

As before, we have $b_0 < p_0$ and $x \in Y$ where $x = z(b_0)$.

Now, we can prove $(1-q)\,x+qy\notin Y$ for all q in $]0,1[$, in contradiction with the hypothesis that Y is weakly-convex. To this end we first remark that $(1-q)\,x+qy=z\,(p_q)$, where $p_q=(1-q)\,b_0+qa_0\in]b_0,a_0[$. There are $a\in A$ and $b\in B$ such that $b_0\le b<p_q<a\le a_0$. If $p_q\ge p_0$, by the definition of A we have $z\,(p_q)\in Z$, and if $p_q<p_0$, by the definition of B we have $z\,(p_q)\in Z$. Hence $(1-q)\,x+qy\notin Y$, and the proof of Theorem 2.1 is achieved. □

Remark 2.2. a) When X is the Euclidean space $\mathbb{R}^n$ with finite dimension n, Theorem 2.1 has been established by V.F. Dem'janov and L.V. Vasil'ev [4], p. 16.

b) Since the above proof uses only the continuity of the function $p\to(1-p)\,x_0+py_0p\in[0,1]$, with fixed x_0 and y, Theorem 2.1 remains valid for topological vector groups of special type (cf. [9]).

c) As Example 1.1 shows, Theorem 2.1 fails for weakly-convex sets which are not closed. □

It is well known that the closure of every convex set in a topological vector space is convex (cf.[2], p.57). The following corollary shows that this result preserves for p-convex sets too.

Corollary 2.3. *The closure of every p-convex set Y in a topological vector spaces in convex.*

Proof. The adherence $\bar{Y}$ is p-convex since

$$(1-p)\,\bar{Y}+p\bar{Y}=\overline{(1-p)\,Y}+p\bar{Y}\subset\overline{(1-p)\,Y+pY}\subset\bar{Y}.$$

By Theorem 2.1, $\bar{Y}$ is convex.

When $p=\frac{1}{2}$ this corollary has been proved by J. von Neumann [11]. □

3. THE CONVEXITY OF p-CONVEX SETS.

As Example 1.2 shows, Theorem 2.1 is false for open sets. However, this theorem remains true for p-convex open sets. In proving this we are in need of the following lemma of J.W. Greem and W. Gustin [6]:

Lemma 3.1. *Let $p\in]0,1[$. Denote by $(P_n)_{n\ge1}$ the sequence of sets inductively defined as follows: $P_1=\{0,p,1\}$: if $P_n=\{0,p_n^{(1)},p_n^{(2)},\ldots,p_n^{(2^n)},1\}$ where $0<p_n^{(1)}<\ldots<p_n^{(2^n)}<1$, has been already defined, put*

$$P_{n+1}=P_n\cup\{(1-p)\,p_n^{(k-1)}+pp_n^{(k)}:1\le k\le 2^n+1\},$$

where $p_n^{(0)}=0$ and $p_n^{(2^n+1)}=1$. Then the set $P=\cup\{P_n:n\ge1\}$ is dense in the interval $[0,1]$.

Lemma 3.2. *If Y is a p-convex set in a vector space, then Y is q-convex for each q in the set P in Lemma 3.1.*

Proof. If suffices to prove that $q\in P_n, n\ge1$, implies

$$(1-q)\,Y+qY\subset Y. \tag{3.1}$$

When $n=1$ and $q\in P_1=\{0,p,1\}$, the inclusion (3.1) is immediate.

Suppose (3.1) is valid for an integer $n \geq 1$. Let $q \in P_{n+1}$. We can admit that $q \notin P_n$ so that q has the form

$$q = (1-p)\, p_n^{(k-1)} + pp_n^{(k)} \text{ for a } k \in \{1,..,2^n+1\}.$$

If $z \in (1-q)\, Y + qY$, hence $z = (1-q)\, x + qy$ with $x, y, \in Y$, then $z = (1-p)\, u + pv$, where

$$u = \left(1 - p_n^{(k-1)}\right) x + p_n^{(k-1)} y \in \left(1 - p_n^{(k-1)}\right) Y + p_n^{(k-1)} Y \subset Y$$

and

$$v = \left(1 - p_n^{(k)}\right) x + p_n^{(k)} y \in Y$$

since $p_n^{(k-1)},\ p_n^{(k)} \in P_n$. Therefore $z = (1-p)\, u + pv \in (1-p)\, Y + pY \subset Y$.

We now state an accessibility result for p-convex sets. □

THEOREM 3.3. *If Y is a p-convex set in a topological vector space X, $x \in intY$, $y \in \bar{Y}$ and $0 < \alpha < 1$, then*

$$(1-\alpha)\, x + \alpha y \in intY. \tag{3.2}$$

Proof. First we prove that if $x' intY$, $y' \in Y$ and $0 < \alpha < 1$, then

$$(1-\alpha)\, x' + \alpha y' \in Y. \tag{3.3}$$

By Lemma 3.2, there is a positive number s in P such that $\alpha < 1 - s$.

Since the function $t \to x' + t \cdot \frac{\alpha}{s}\,(y' - x')\,, t \in [0,1]$, is continuous at $t = 0$ and Y is a neighbourhood of x', there exists a $\gamma \leq \frac{\alpha}{s}$, $s < \gamma \leq 1$, such that

$$x' + t \cdot \tfrac{\alpha}{s} gg \cdot s\left(y' - x'\right) \in Y \text{ for all } \left[0, \tfrac{\gamma \cdot x}{\alpha}\right]. \tag{3.4}$$

There is a t_0 in $]0, \frac{\gamma \cdot s}{\alpha}[$ such that

$$(1 \to \delta_{t_0})\, x' + \delta_{t_0} y' \in Y \text{ where } \delta_{t_0} = (1 - t_0)\, \tfrac{\alpha}{1-s}. \tag{3.5}$$

Indeed, the interval $I = \left[\left(1 - \frac{\gamma \cdot s}{\alpha}\right) \frac{\alpha}{1-s}, \frac{\alpha}{1-s}\right] \subset [0,1]$ has a positive length hence by Lemma 3.1, there exists $q \in P \cap int\ I$. It follows that one can find a $T_0 \in]0, \frac{\gamma \cdot s}{\alpha}[$ with $q = \delta_{t_0} \in P$ and, by Lemma 3.2, we obtain $(1 - \delta_{t_0})\, x' + \delta_{t_0} y' \in Y$.

From (3.4) and (3.5) we deduce $(1-r)\, x' + ry' \in Y$, where $r = \frac{\alpha}{s} \cdot t_0$, and

$$(1-\alpha)\, x' + xy' = (1-s)\left[(1 - \delta_{t_0})\, x' + \delta_{t_0} y'\right] + s\left[(1-r)\, x' + ry'\right] \in Y$$

because Y is s-convex. Thus (3.3) is proved.

Now, we are in a position to prove (3.2). Denote $z = (1-\alpha)\, x + xy$. Since the function $f : X \to X$ given by

$$f(u) = u + \tfrac{1}{1-\alpha}\,(z - u)$$

is continuous at $u = y$ and $f(y) = x$, there exists a neighbourhood V of y such that $f(V) \subset int\ Y$. There is a $y' \in Y \cap V$ because $y \in \bar{Y}$. From $f(y') \in f(V) \subset int\ Y$ and (3.3) we derive $z = (1-\alpha)\, f(y') + \alpha y' \in int\ Y$. This proves the theorem.

It is well known that the interior of a convex set in a topological vector space is convex (cf. [2], p.55). The following corollary shows that this result holds even for p-convex sets. □

Corollary 3.4. *The interior of every p-convex in a topological vector space X is convex.*

When $X = \mathbb{C}$ Corollary 3.4 has been established by D.A. Horowitz, D.A. Rose and E.B. Saff [7].

4. THE CONVEXITY OF WEAKLY-CONVEX FUNCTIONS

The epigraph of a real function f defined on a set Y is the set

$$E_f = \{(x, z) \in Y \times \mathbb{R} : f(x) \leq z\}.$$

We need the following well-known lemmas (cf. [1], pp.75-76):

Lemma 4.1. *A real function defined on a convex set in a vector space is convex if and only if its epigraph is convex.*

Lemma 4.2. *If a real function f is lower-semicontinuous on a topological space X, then its epigraph is closed in the topological product $X \times \mathbb{R}$.*

Theorem 4.3. *Let f be a real function defined on a closed set Y in a topological vector space. If f is weakly convex and lower-semicontinuous, then f is convex.*

Proof. First we prove that the epigraph E_f is weakly-convex. Let $(x_1, z_1) \in E_f$ and $(x_2, z_2) \in E_f$. Since f is weakly-convex, there exists a p in $]0, 1[$ such that $(1-p)\, x_1 + px_2 \in Y$ and $f((1-p)\, x_1 + px_2) \leq (1-p)\, f(x_1) + pf(x_2) \leq (1-p)\, z_1 + pz_2$, hence $((1-p)\, x_1 + px_2, (1-p)\, z_1 + pz_2) \in E_f$.

Now, by Lemma 4.2 and Theorem 2.1, Y and E_f are convex, hence, by Lemma 4.1, f is convex. □

Corollary 4.4. *Let f be a real function defined on a closed convex set in a topological vector space. If f is wealky-convex and lower-semicontinuous, then f is convex.*

Remark 4.5. When the weakly-convexity is replaced by the stronger condition of p-convexity and the lower-semicontinuous is weakened in different ways, Corollary 4.4 remains still true. More precisely, the convexity of every real function f which is $\frac{1}{2}$ - convex on an interval in $\mathbb{R}$ has been established by A. Ostrowski [12] when f is bounded on a set of positive measure, and by M.R. Mehdi [8] when f is bounded on a set second category having the Baire property. An extension of last results to p-convex functions has been given by E. Deák [3].

However, as the following example shows, the results of A. Ostrowski and M.R. Mehdi fail when the considered functions are only weakly-convex. □

Example 4.6. The function $f : \mathbb{R} \to \mathbb{R}$, defined buy $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational, satisfies the conditions of A. Ostrowski and

M.D. Mehdi and it is weakly-convex. We shall show that f is not p-convex for any p in $]0,1[$. Supposing the contrary, there exists a p in $]0,1[$ such that for every $x, y \in \mathbb{R}$ the inequality (1.2) holds. If p is irrational, use (1.2) with $x = \frac{1}{1-p}$ and $y = \frac{1}{p}$ to arrive at the contradiction

$$1 = f(2) = f((1-p)x + py) \leq 1 - pf(x) + pf(y) = 0.$$

If p is rational, use (1.2) with $x = \sqrt{2}$ and $y = \frac{p-1}{p}\sqrt{2}$ to arrive at the contradiction.

$$1 = f(0) = f((1-p)x + py) \leq (1-p) f(x) + pf(y) = 0.$$

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