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ON APPROXIMATION BY LINEAR OPERATORS: IMPROVED ESTIMATES

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Abstract. The present paper describes a unified approach to quantitative approximation theorems for certain linear operators L including positive linear ones. It is shown for so-called almost lattice homomorphisms A that the difference $(L - A)(f, x)$ can be estimated in terms of a certain three parameter functional Ω. This functional is in turn bounded from above by various classical seminorms such as (modifications of) moduli of continuity of order 1 and 2. There is a large variety of opportunities to combine results of this paper in order to arrive at direct quantitative assertions. Several examples show that the general theory implies a number of results which improve those known so far.

I. Introduction. The aim of the present paper is to study the degree of approximation of continuous functions by certain linear operators including positive linear ones. Very much has been done in this field especially as far as direct theorems are concerned. A partial survey of the corresponding literature is given in a forthcoming bibliography on Bernstein type operators [9]; these are heavily used as examples when treating positive linear operators (*PLO*'s) in general.

One of the observations that we made while working on the subject was that almost all of the authors were basically using the same machinery. This boiled down to the question of how to describe the underlying idea more closely by giving one single estimate which in turn implies most of the inequalities with 'classical' right-hand sides such as moduli of continuity of different orders. This is part of what the present paper intends. Another question was how to obtain improvements over the known results; this is due to the well-known fact that the constants and sometimes even the order which are obtained after applying a quantitative Korovkin type theorem are poor. Answers to this question can also be derived from the present paper.

As far as the underlying mathematical technique is concerned, all our estimates are based upon the use of a new K-functional which will be described in Section II. Moreover, we have made sure that all our subsequent estimates involve a great degree of freedom in the sense that when applied to some concrete operators the choice of the parameters appearing in the moduli of continuity is up to the user. This is the most convenient way to compare our results to what is available in the literature.

Although we only treat the case of approximation of continuous functions on a compact segment $[a, b]$ it is possible to extend these results to the approximation of functions in several variables by using, for instance, tensor product methods. Thus the main idea of this paper is the fact that the approximation behaviour of a sequence of certain operators mapping $C[a, b]$ into $C[c, d]$, $[c, d] \subseteq [a, b]$, can be described by using one single functional which can be estimated from above by a variety of more classical seminorms such as moduli of continuity. For the sake of brevity our paper is split into five parts. In Section II the new K-functional Ω is introduced. Section III establishes relationships between Ω and some moduli of continuity. In Section IV we give some estimates for the approximation of so-called almost lattice homomorphisms by certain linear operators in terms of Ω. The combination of the results presented in Sections III and IV then turns out to be an effective tool for proving estimates in terms of the more classical quantities mentioned above. This becomes clear in Section V where we apply our results to approximation by Bernstein operators, by operators of Meyer-König and Zeller and by some non-positive Hermite-Fejér operators. It has to be noted, however, that the underlying technique works in many other cases as well.

Throughout this paper $[a, b]$ is a finite interval of the real axis. For $r \geq 0$ the symbol $C^r = C^r[a, b]$ denotes the space of r-times continuously differentiable functions. Moreover, $C := C^0 \cdot \|f^{(r)}\|$ denotes the sup-norm of the r-th derivative of a function $f \in C^r \cdot e_i$ is always the i-th monomial given by $e_i(x) := x^i$, $x \in [a, b]$, $i \geq 0$.

II. The functional Ω. A standard technique for the study of rates of approximation of continuous functions f in terms of moduli of smoothness is based on the use of

$$K_r(f, t) = \inf \{\|f - g\| + t \cdot \|g^{(r)}\| : g \in C^r\}.$$

This method is described in an excellent survey paper of R.A. DeVore [5]. As can be seen from the definition of $K_r(f, t)$, its value expresses how small the norm of $f - g$ can be made compared to the norm of $t \cdot g^{(r)}$.

In the present paper we use a special case of the functional $\Omega_{r,r+s}$ given for $r, s \geq 1$ and $(f; t, t_r, t_{r+s}) \in C \times \mathbb{R}_+^3$ by the expression

$$\Omega_{r,r+s}(f; t, t_r, t_{r+s}) :=$$
$$\inf \{\|f - f_r\| + t \cdot \inf(\|f_r^{(r)} - f_{r+s}^{(r)}\| + t_r \cdot \|f_{r+s}^{(r)}\| + t_{r+s} \cdot \|f_{r+s}^{(r+s)}\| : f_{r+s} \in C^{r+s}) : f_r \in C^r\}.$$

$\Omega_{r,r+s}$ expresses two things. First it does the same as K_r as can be seen from the inequality

$$\Omega_{r,r+s}(f; t, t_r, t_{r+s}) \leq K_r(f, t).$$

Secondly, it controls how smooth the smoothing functions are themselves. For instance, for $f \in C^r[a, b]$ we have

$$\Omega_{r,r+s}(f; t, t_r, t_{r+s}) \leq$$
$$\leq t \cdot \inf(\|f^{(r)} - f_{r+s}^{(r)}\| + t_r \cdot \|f_{r+s}^{(r)}\| + t_{r+s} \cdot \|f_{r+s}^{(r+s)}\| : f_{r+s} \in C^{r+s}),$$

where the right-hand side is basically a slight modification of K_r for r-times differentiable functions with respect to such in C^{r+s}.

In the sequel we shall only use $\Omega := \Omega_{1,2}$. In order to simplify some of our below considerations, we shall describe Ω in a different way. For this purpose let E denote a real vector space, U a subspace of E and p and $\bar{p}$ seminorms on E and U, respectively. We define $\tilde{K} : \mathbb{R}_+^2 \times E \to \mathbb{R}_+$ by

$$\tilde{K}(t_1, t_2, f; (E, p), (U, \bar{p})) := \inf \{p(f - g) + t_1 p(g) + t_2 \bar{p}(g) : g \in U\}$$

and $K : \mathbb{R}_+ \times E \to \mathbb{R}_+$ by the equation

$$K(t, f; (E, p), (U, \bar{p})) := \inf \{p(f - g) + t \cdot \bar{p}(g) : g \in U\}.$$

We write for simplification $\tilde{K}(t_1, t_2, f)$ and $K(t, f)$, respectively, if it is clear what (E, p) and $(U, \bar{p})$ are. It is easy to prove that for fixed $(t_1, t_2) \in \mathbb{R}_+^2$ the functional $\tilde{K}(t_1, t_2, \cdot)$ is a seminorm on E; thus we can use it as the seminorm $\bar{p}$ when defining a functional $K(t, \cdot)$ for $t \in \mathbb{R}_+$.

We now consider the spaces C^i, $i \in \{0, 1, 2\}$, of i-times continuously differentiable functions defined on $[a, b]$ with the seminorms $\|\cdot^{(i)}\|$. Here $\cdot^{(i)}$ denotes i-fold differentiation. Then, using the seminorm

$$\tilde{K}(t_1, t_2, f; (C^1, \|\cdot^{(1)}\|), (C^2, \|\cdot^{(2)}\|)) \text{ on } C^1,$$

we have

$$\Omega(f; t, t_1, t_2) = K(t, f; (C, \|\cdot\|), (C^1, \tilde{K}(t_1, t_2, (\cdot)))).$$

In particular, Ω is a seminorm on C depending upon the three parameters t, t_1, t_2. These three parameters will be used later to describe approximation properties of certain linear operators.

III. The Relationship between Ω and Some Moduli of Continuity. In this section Ω will be estimated from above by more classical quantities such as moduli of continuity. All majorants of Ω will contain a 'free variable' $h > 0$ which makes it more convenient to find clear majorants in concrete examples.

We first consider the case of continuously differentiable functions since this will be used below. Natural majorants for Ω are in this case expressions involving the first order modulus of continuity of f', defined by $\omega_1(f', h) := \sup \{|f'(x) - f'(y)| : x, y \in [a, b], |x - y| \leq h\}$, or the east concave majorant $\tilde{\omega}_1(f', \cdot)$ of $\omega_1(f', \cdot)$ given by

$$\tilde{\omega}_1(f', h) = \sup \left\{ \sum_{i=1}^n \lambda_i \omega_1(f', h_i) : n \in \mathbb{N}, \sum_{i=1}^n \lambda_i = 1, \sum_{i=1}^n \lambda_i h_i = h, \lambda_i \geq 0 \right\}, \; h \geq 0.$$

Both quantities are related by $\omega_1(f', \cdot) \leq \tilde{\omega}_1(f', \cdot) \leq 2 \cdot \omega_1(f', \cdot)$.

THEOREM 3.1 *Let Ω be defined as above. If $(f; t, t_1, t_2) \in C^1[a,b] \times \mathbb{R}_+^3$ is arbitrarily given, then the following are true for any $h > 0$:*

$$\Omega(f:t,t_1,t_2) \le \begin{cases} (i) & t\cdot\left\{\min(1,t_1)\cdot\|f'\| + \chi_{[0,1]}(t_1)\cdot\max\left(\frac{1+t_1}{2}, \frac{t_2}{h}\right)\cdot\tilde{\omega}_1(f',h)\right\}, \\ (ii) & t\cdot\left\{t_1\cdot\|f'\| + \left[1+\frac{t_2}{h}\cdot\max\left(0, 1-\frac{h}{b-a}\right)\right]\cdot\omega_1(f',h)\right\}. \end{cases}$$

Here $\chi_{[0,1]}$ is the characteristic function of $[0,1]$.

COROLLARY 3.2 *For $t_1 = 0$ the inequalities from above imply:*

$$\Omega(f;t,0,t_2) \le \begin{cases} (i) & t\cdot\max\left(\frac{1}{2}, \frac{t_2}{h}\right)\cdot\tilde{\omega}_1(f',h), \\ (ii) & t\cdot\left[1+\frac{t_2}{h}\cdot\max\left(0,1-\frac{h}{b-a}\right)\right]\cdot\omega_1(f',h), \\ (iii) & t\cdot\min\left[\max\left(1,\frac{2t_2}{h}\right), 1+\frac{t_2}{h}\right]\omega_1(f',h). \end{cases}$$

Proof of Theorem 3.1. If f is continuously differentiable, then the definition of Ω, being a special K-functional, yields

$$\Omega(f;t,t_1,t_2) \le t\cdot\tilde{K}(t_1,t_2,f;(C^{(1)},\|\cdot^{(1)}\|),(C^{(2)},\|\cdot^{(2)}\|)).$$

Several properties of the functionals $\tilde{K}$ were investigated in [6]. In particular, it was shown that two functionals $\tilde{K}$ and K constructed with the aid of the same pairs (E,p) and $(U,\bar{p})$ are related by the inequality $(t_1, t_2 \ge 0)$ $\tilde{K}(t_1,t_2,f;(E,p),(U,\bar{p}))$

$$\le \min(1,t_1)\cdot p(f) + (1+t_1)\cdot\chi_{[0,1]}(t_1)\cdot K\left(\frac{t_2}{1+t_1}, f; (E,p),(U,\bar{p})\right).$$

Since for $h>0$ the functional K satisfies the inequality

$$K\left(\frac{t_2}{1+t_1}\cdot\frac{2}{h}\cdot\frac{h}{2}, f; (E,p),(U,\bar{p})\right) \le$$

$$\le \max\left(1,\frac{2t_2}{(1+t_1)h}\right)\cdot K\left(\frac{h}{2}, f; (E,p),(U,\bar{p})\right),$$

we arrive at

$$\Omega(f;t,t_1,t_2) \le t\cdot\left\{\min(1,t_1)\cdot p(f) + \chi_{[0,1]}(t_1)\cdot\max\left(1+t_1, \frac{2t_2}{h}\right)\cdot\right.$$

$$\left.\cdot K\left(\frac{h}{2}, f; (E,p),(U,\bar{p})\right)\right\}.$$

Putting $(E,p) = (C^1, \|\cdot^{(1)}\|)$, $(U,\bar{p}) = (C^2, \|\cdot^{(2)}\|)$, it is known that (see J. Peetre [14], B. S. Mitjagin/E. M. Semenov [13])

$$K\left(\frac{h}{2}, f; (E,p),(U,\bar{p})\right) = \frac{1}{2}\cdot\tilde{\omega}_1(f',h).$$

Combining this with our observations from above now leads to

$$\Omega(f;t,t_1,t_2) \le t\cdot\left\{\min(1,t_1)\cdot\|f'\| + \chi_{[0,1]}(t_1)\cdot\max\left(\frac{1+t_1}{2}, \frac{t_2}{h}\right)\cdot\tilde{\omega}_1(f',h)\right\}.$$

This was the claim in (i).
For the proof of (ii) we use again the fact that

$$\Omega(f;t,t_1,t_2) \le t\cdot\inf(\|f'-f_2'\| + t_1\cdot\|f_2'\| + t_2\|f_2''\| : f_2 \in C^2).$$

Assume first that $0 < h \le b-a$, and let f_2 be any primitive of $U_h(f',\cdot)$, given by

$$U_h(f',x) := \frac{1}{h}\int_{-h\left(\frac{x-a}{b-a}\right)}^{-h\left(\frac{x-a}{b-a}\right)+h} f'(x+t)\,dt.$$

As was shown by V. V. Žuk and G. I. Natanson [22], we have

$$|f'(x) - f_2'(x)| = |f'(x) - U_h(f',x)| \le \omega_1(f',h),$$

$$\|f_2'\| = \|U_h(f',\cdot)\| \le \|f'\|, \text{ and } \|f_2''\| = \|(U_h(f',\cdot))'\| \le$$

$$\le \frac{1}{h}\left(1-\frac{h}{b-a}\right)\cdot\omega_1(f',h).$$

For $h \ge b-a$ we put $U_h := U_{b-a}$. In this case

$$|f'(x) - f_2(x)| \le \omega_1(f', b-a) = \omega_1(f',h) \text{ and } f_2'' = 0.$$

This shows the validity of (ii). ∎

Proof of Corollary 3.2. (i) and (ii) are immediate consequences of the corresponding inequalities in Theorem 3.1. (iii) is a simple consequence of (i) and (ii), if one observes that $\tilde{\omega}_1(f',\cdot) \le 2\cdot\omega_1(f_1',\cdot)$ and $\max\left(0, 1-\frac{h}{b-a}\right) \le 1$. ∎

Our next theorem contains estimates of $\Omega(f;t,t_1,t_2)$ for arbitrary continuous functions defined on $[a,b]$. These are given in terms of the usual first order modulus of continuity ω_1, a modification of this modulus of continuity introduced by M. Marsden and I. J. Schoenberg [11] and denoted by ω_1^*, and in terms of the least concave majorant of the first order modulus notated by $\tilde{\omega}_1$.

THEOREM 3.3 *Let Ω be defined as above. If $(f;\ t,\ t_1,\ t_2) \in C[a,b]\times \mathbb{R}^3_+$ is arbitrarily given, then for any $h>0$ the following inequalities hold:*

$$\Omega(f;\ t, t_1, t_2) \le \begin{cases} (i) & \dfrac{1}{2}\cdot\max\left(1, \dfrac{2t}{h}\right)\cdot\tilde{\omega}_1(f,\ h), \\ (ii) & \inf\left\{\left[1+\dfrac{t}{h}\cdot\max\left(0, 1-\dfrac{h}{b-a}\right)\right]\cdot\omega_1(f-c\cdot e_1, h)+t\cdot t_1\cdot|c| : c\in\mathbb{R}\right\}, \\ (iii) & \min\left[\max\left(1, \dfrac{2t}{h}\right), 1+\dfrac{t}{h}\right]\cdot\omega_1(f,\ h). \end{cases}$$

An immediate consequence is

COROLLARY 3.4

$$\Omega(f;\ t,\ 0,\ t_2) \le \left[1+\frac{t}{h}\cdot\max\left(0, 1-\frac{h}{b-a}\right)\right]\cdot\omega_1^*(f,\ h),$$

where $\omega_1^*(f,\ h) := \inf\{\omega_1(f-c\cdot e_1,\ h) : c\in\mathbb{R}\}$.

Proof of Theorem 3.3. In order to arrive at (i) we proceed as in the corresponding part of the proof of Theorem 3.1 observing in this case that

$$\Omega(f;\ t,\ t_1,\ t_2) \le K(t, f; (C, \|\cdot\|), (C^1, \|\cdot^{(1)}\|))$$

$$\le \max\left(1, \frac{2t}{h}\right)\cdot K\left(\frac{h}{2}, f;\ C,\ C^1\right) = \max\left(1, \frac{2t}{h}\right)\cdot\frac{1}{2}\cdot\tilde{\omega}_1(f,\ h),$$

which follows again from the work of B. S. Mitjagin and E. M. Semenov, for instance.

(ii) is obtained as follows. For a fixed triple $(t,\ t_1,\ t_2)$ the functional $\Omega(\cdot\,;\ t,\ t_1,\ t_2)$ is a seminorm on C. Thus, if l is any linear function, then

$$\Omega(f;\ t,\ t_1,\ t_2) \le \Omega(f-l;\ t,\ t_1,\ t_2) + \Omega(l;\ t,\ t_1,\ t_2).$$

An estimate for the first term on the right-hand side can be given by using the idea of Žuk and Natanson again. Thus

$$\Omega(f-l;\ t,\ t_1,\ t_2) \le \left[1+\frac{t}{h}\cdot\max\left(0, 1-\frac{h}{b-a}\right)\right]\cdot\omega_1(f-l,\ h).$$

Since l is continuously differentiable, it follows from Theorem 3.1 that

$$\Omega(l;\ t,\ t_1,\ t_2) \le t\cdot t_1\cdot\|l'\|.$$

Thus, if l is given by $l = c\cdot e_1 + d\cdot e_0$, then the combination of both inequalities leads to

$$\Omega(f;\ t, t_1, t_2) \le \left[1+\frac{t}{h}\cdot\max\left(0, 1-\frac{h}{b-a}\right)\right]\cdot\omega_1(f-c\cdot e_1,\ h)+t\cdot t_1\cdot|c|.$$

Passing to the infimum over all such numbers gives the estimate in (ii). (iii) Substituting $c = 0$ into (ii) gives the majorant

$$\left[1+\frac{t}{h}\cdot\max\left(0, 1-\frac{h}{b-a}\right)\right]\cdot\omega_1(f,\ h) \le \left(1+\frac{t}{h}\right)\cdot\omega_1(f,\ h).$$

Moreover, taking the inequality $\tilde{\omega}_1(f,\ h) \le 2\cdot\omega_1(f,\ h)$ into account shows that the right hand side of (i) can be estimated from above by $\max\left(1, \dfrac{2t}{h}\right)\cdot\omega_1(f,\ h)$. A combination of these observations gives (iii). ∎

The inequality in Corollary 3.4 is a trivial consequence of inequality (ii) in Theorem 3.3.

The following theorem provides us with an estimate for $\Omega(f;\ t,\ t_1,\ t_2)$ using mainly the second order modulus of continuity $\omega_2(f,\ \cdot)$ of an $f\in C$ and again involving a 'free variable' $h>0$. Here $\omega_2(f,\ \cdot)$ is given by

$$\omega_2(f,\ h) := \sup\{|f(x-\delta) - 2f(x) + f(x+\delta)| : x, x\pm\delta\in[a,b], 0<\delta\le h\}.$$

THEOREM 3.5 *Let Ω be defined as above. If $(f;\ t,\ t_1,\ t_2)\in C[a,b]\times\mathbb{R}^3_+$ is arbitrarily given, then for any $h>0$ the following inequality holds:*

$$\Omega(f;\ t,\ t_1,\ t_2) \le \left[\frac{3}{2}+2tt_2\cdot\max\left(\frac{1}{h^2}, \frac{1}{(b-a)^2}\right)\right]\omega_2(f,\ h) + \left[2tt_1\cdot\max\left(\frac{1}{h}, \frac{1}{b-a}\right)\right]\cdot\omega_1(f,\ h).$$

An immediate consequence is

COROLLARY 3.6 *If $t_1 = 0$, then the above estimate reduces to*

$$\Omega(f;\ t,\ 0,\ t_2) \le \left[\frac{3}{2}+2tt_2\cdot\max\left(\frac{1}{h^2}, \frac{1}{(b-a)^2}\right)\right]\cdot\omega_2(f,\ h).$$

Proof of Theorem 3.5. From the definition of Ω it follows that

$$\Omega(f;\ t,\ t_1,\ t_2) \le \inf\{\|f-f_1\| + t\cdot(t_1\cdot\|f_1'\| + t_2\cdot\|f_1''\|) : f_1\in C^2\}.$$

Thus it remains to show that for any $h>0$ there is an $f_1 = f_h\in C^2$ which yields the estimate claimed above. The idea used to obtain such a function is different from the one used in Theorems 3.1 and 3.3. First we define a suitable extension of $f\in C[a,b]$ and then use Steklov means with respect to this extension in order to obtain a smooth approximation of f which in turn satisfies the necessary inequalities. It is advantageous for our purposes to define an extension which itself depends upon h. Assume first that $0<h\le b-a$. We define

$f_h : [a-h,\ b+h]\to\mathbb{R}$ by

$$f_h(x) := \begin{cases} f(x+h) - f(a+h) + f(a) & \text{for } x\in[a-h,\ a], \\ f(x) & \text{for } x\in[a,\ b], \\ f(x-h) - f(b-h) + f(b) & \text{for } x\in[b,\ b+h]. \end{cases}$$

Then $\tilde{f}_h(x)$ is well defined and continuous.
For $x \in [a, b]$ we consider differences of the form

$$\tilde{f}_h(x + r) - 2\tilde{f}_h(x) + \tilde{f}_h(x - r), \quad 0 < r \le h.$$

(1) $x + r, x, x - r \in [a, b]$. In this case the absolute value of the above expression is equal to

$$|f(x + r) - 2f(x) + f(x - r)| \le \omega_2(f, h).$$

(2) $x + r, x \in [a, b]$, $x - r < a$ and $x < \frac{a+b}{2}$. Then

$$\tilde{f}_h(x + r) - 2\tilde{f}_h(x) + \tilde{f}_h(x - r)$$
$$= f(a) - 2f(x) + f(2x - a) + f(x + r) - 2f\left(x + \frac{h}{2}\right) +$$
$$+ f(x - r + h) - f(2x - a) + 2f\left(x + \frac{h}{2}\right) - f(a + h).$$

This implies

$$|\tilde{f}_h(x + r) - 2\tilde{f}_h(x) + \tilde{f}_h(x - r)|$$
$$\le \omega_2(f, |x - a|) + \omega_2\left(f, \left|-r + \frac{h}{2}\right|\right) + \omega_2\left(f, \left|a - x + \frac{h}{2}\right|\right)$$
$$\le \omega_2(f, r) + \omega_2(f, h/2) + \omega_2(f, h/2),$$

since $0 < r \le h$, $a - x < r$ and $a - x < 0$.

(3) The case $x + r, x \in [a, b]$, $x - r < a$ and $x \ge \frac{a+b}{2}$ cannot occur since this would be a contradiction to the condition $x + r \le b$.

(4) We consider now the particular case where $r = h$. Under the assumptions $x + h, x \in [a, b]$, $x - h < a$ we obtain the inequalities

$$|\tilde{f}_h(x + h) - 2\tilde{f}_h(x) + \tilde{f}_h(x - h)|$$
$$= \left|f(x + h) - 2f\left(\frac{x + h + a}{2}\right) + f(a) - f(x) + 2f\left(\frac{x + h + a}{2}\right) - f(a + h)\right|$$
$$\le \omega_2\left(f, \frac{1}{2}|a - x - h|\right) + \omega_2\left(f, \frac{1}{2}|a + h - x|\right)$$
$$\le \omega_2(f, h) + \omega_2(f, h/2).$$

(5) $x - r, x \in [a, b]$, $x + r > b$ and $x > \frac{a+b}{2}$. This case is analogous to the one in (2).

(6) The case $x - r, x \in [a, b]$, $x + r > b$ and $x \le \frac{a+b}{2}$ cannot occur (cf. case (3)).

(7) The special case $r = h$, $x - h$, $x \in [a, b]$, $x + h > b$ is similar to t
one under (4).

(8) $x - r < a$, $x + r > b$. Obviously this implies $r, h > \frac{b-a}{2}$. H
we have

$$|\tilde{f}_h(x + r) - 2\tilde{f}_h(x) + \tilde{f}_h(x - r)|$$
$$= \Big|f(x - r + h) - 2f(x) + f(x + r - h) - f(b - h) +$$
$$+ 2f\left(\frac{a+b}{2}\right) - f(a + h) + f(b) - 2f\left(\frac{a+b}{2}\right) + f(a)\Big|$$
$$\le \omega_2(f, |r - h|) + \omega_2\left(f, \frac{1}{2}|b - a - 2h|\right) + \omega_2\left(f, \frac{b-a}{2}\right)$$
$$\le \omega_2\left(f, \frac{h}{2}\right) + 2\omega_2(f, h).$$

(9) For the special case $r = h$ the assumptions $x - h < a$, $x \in [a,$
$x + h > b$ $\left(\text{i.e. } h > \frac{b-a}{2}\right)$ lead to the inequality

$$|\tilde{f}_h(x + h) - 2\tilde{f}_h(x) + \tilde{f}_h(x - h)|$$
$$= \left|-f(b - h) + 2f\left(\frac{a+b}{2}\right) - f(a + h) + f(b) - 2f\left(\frac{a+b}{2}\right) + f(a)\right.$$
$$\le \omega_2\left(f, \left|\frac{a+b}{2} - b + h\right|\right) + \omega_2\left(f, \frac{b-a}{2}\right) \le \omega_2\left(f, \frac{h}{2}\right) + \omega_2(f, h$$

The observations made under (1),...,(9) will now be used to constr
the functions that we were looking for. For $0 < h \le b - a$ the secc
order Steklov mean of $f \in C[a, b]$ (with respect to the extension $\tilde{f}_h$) is def
ed by

$$f_h(x) := \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \tilde{f}_h(x + s + t)\, ds\, dt, \quad x \in [a, b].$$

Because of

$$f_h(x) = \frac{1}{2} \cdot \frac{1}{h^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} [\tilde{f}_h(x + s + t) + \tilde{f}_h(x - s - t)]\, ds\, dt$$

we have

$$|f(x) - f_h(x)| \le \frac{1}{2}\cdot\frac{1}{h^2}\cdot\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} |\tilde{f}_h(x+s+t) - 2f(x) + \tilde{f}_h(x-s-t)|\, ds\, dt$$

$$\le \frac{1}{2}\left(\omega_2\left(f, \frac{h}{2}\right) + 2\omega_2(f, h)\right);$$

the last inequality following from the observations made under (1) — (9). Moreover, the second derivative of the second order Steklov mean satisfies

$$|f_h''(x)| = \frac{1}{h^2}\cdot|\tilde{f}_h(x+h) - 2\tilde{f}_h(x) + \tilde{f}_h(x-h)|$$

$$\le \frac{1}{h^2}\cdot\left(\omega_2\left(f, \frac{h}{2}\right) + \omega_2(f, h)\right);$$

this follows from (4), (7), and (9).
The first derivative of the second order Steklov means can be estimated as follows:

$$|f_h'(x)| = \frac{1}{h^2}\cdot\left|\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[\tilde{f}_h\left(x+t+\frac{h}{2}\right) - \tilde{f}_h\left(x+t-\frac{h}{2}\right)\right] dt\right|$$

$$\le \frac{1}{h^2}\cdot\int_{-\frac{h}{2}}^{\frac{h}{2}} \omega_1(\tilde{f}_h, h)\, dt = \frac{1}{h}\cdot\omega_1(\tilde{f}_h, h) \le \frac{2}{h}\cdot\omega_1(f, h).$$

We substitute the estimates for $\|f - f_h\|$, $\|f_h''\|$, and $\|f_h'\|$ now into the majorant for $\Omega(f; t, t_1, t_2)$ given at the beginning of this proof, replacing $f_1 \in C^2[a, b]$ by the second order Steklov means constructed above. This shows that for $0 < h \le b - a$ the inequalities

$$\Omega(f; t, t_1, t_2) \le \|f - f_h\| + t\cdot(t_1\cdot\|f_h''\| + t_2\cdot\|f_h''\|)$$

$$\le \left(\frac{3}{2} + 2t\cdot t_2\cdot\frac{1}{h^2}\right)\cdot\omega_2(f, h) + 2t\cdot t_1\cdot\frac{1}{h}\cdot\omega_1(f, h)$$

hold which proves the assertion in Theorem 3.5 for the cases where $0 < h \le b - a$. For $h > b - a$ the Steklov means were not defined since in this case the estimates concerning the second order differences of the above extension $\tilde{f}_h$ cannot be proved in the way we did it.

Therefore for $h > b - a$ we define $f_h := f_{b-a}$.

In this case we have the inequalities

$$\|f - f_h\|_\infty = \|f - f_{b-a}\|_\infty \le \frac{3}{2}\cdot\omega_2(f, b-a) \le \frac{3}{2}\cdot\omega_2(f, h),$$

$$\|f_h''\|_\infty = \|f_{b-a}''\|_\infty \le \frac{2}{(b-a)^2}\cdot\omega_2(f, b-a) \le \frac{2}{(b-a)^2}\cdot\omega_2(f, h),$$

$$\|f_h'\|_\infty = \|f_{b-a}'\|_\infty \le \frac{2}{b-a}\cdot\omega_1(f, b-a) \le \frac{2}{b-a}\,\omega_1(f, h).$$

Combining these inequalities with the ones proved before for $0 < h \le b - a$ we arrive at the estimate claimed in Theorem 3.5. ∎

Remark. In view of Theorems 3.3 and 3.5 it is natural to ask what the relationship between ω_1^* and ω_2 is since both annihilate linear functions. It has to be noted that there is no constant $c > 0$ such that $\omega_1^*(f, \delta) \le c\cdot\omega_2(f, \delta)$ for all f and $\delta > 0$. This can be seen from looking at $f(x) = x^2$ on $[-1,1]$, for instance. However, it is possible to show that there exists a constant $d > 0$ satisfying $\omega_1^*(f, \delta) \le d\cdot\omega_2(f, \delta^{1/2})$ for $f \in C[a, b]$ and $\delta > 0$.

IV. Estimates for the Approximation of Almost Lattice Homomorphisms by Certain Linear Operators. Here we present some estimates for the approximation of so-called almost lattice homomorphisms (ALH) by certain linear operators in terms of the functional Ω introduced in Section II. As is clear from the last section such estimates can immediately be turned into inequalities containing different kinds of moduli of continuity. Examples will be given in Section V.

DEFINITION 4.1 *If Y is a non-empty set, $B(Y)$ is the Banach space of real-valued bounded functions on Y and if $C[a, b]$ denotes the space of real-valued continuous functions on an interval $[a, b]$ with $a < b$, then an operator $A : C[a, b] \to B(Y)$ such that*

$$A(f, y) = \psi_A(y)\cdot f(g_A(y)),\ \psi_A \in B(Y),\ g_A : Y \to [a, b],\ f \in C[a, b],\ y \in Y,$$

is called an almost lattice homomorphism (ALH).

The ALH's include all lattice homomorphisms $T : C[a, b] \to C(Y)$, where Y is a compact topological space, as can be seen from the Wolff representation theorem in [21].

The starting point for our investigation is the following observation.

PROPOSITION 4.2 *If A is an ALH given by $A(f, y) = \psi_A(y)\cdot f(g_A(y))$, and if L is an operator, both mapping $C[a, b]$ into $B(Y)$, then for all $f \in C[a, b]$, $y \in Y$, the inequality*

$$|L(f, y) - A(f, y)| \le |L(f, y) - L(e_0, y)\cdot f(g_A(y))| + |L(e_0, y) - A(e_0, y)|\cdot|f(g_A(y))|\ \textit{holds.}$$

The main purpose of the above proposition is to shift the problem of finding a good majorant of $|(L - A)(f, y)|$ to two similar ones. The first of these is to find a majorant for the difference $|(L-\tilde{A})(f,y)|$ where $\tilde{A}$ is given by $\tilde{A}(f,y) = L(e_0,y)\cdot f(g_A(y))$. $\tilde{A}: C[a,b] \to B(Y)$ is a mapping satisfying the equation $L(e_0,y) = \tilde{A}(e_0,y)$ for all y in Y. The second resulting problem is to find a majorant of $|(L-A)(e_0,y)|$ where A denotes the initial ALH. Thus our approach follows the classical pattern of estimating the difference between a *PLO* and the identity (canonical embedding) as for instance employed by R. A. DeVore [4].

Our next theorem generalizes a result given in [7].

THEOREM 4.3. *Let $L: C[a,b] \to B(Y)$ be a continuous linear operator and $A \neq 0$ as above such that the pair (L, A) satisfies the inequalities*

(i) $|(L-A)(f_1,y)| \leq \Phi(y)\cdot\|f_1'\|$ *for all* $f_1 \in C^1[a,b]$ *with a function* $\Phi \geq 0$ *and*

(ii) $|(L-A)(f_2,y)| \leq \gamma_1(y)\cdot\|f_2'\| + \gamma_2(y)\cdot\|f_2''\|$ *for all* $f_2 \in C^2[a,b]$ *with real valued functions* $\gamma_1, \gamma_2 \geq 0$.

Moreover, we assume for all $y \in Y$ that the quotients $\gamma_i(y)/\Phi(y)$, $i = 1, 2$, are finite. Then for every $f \in C[a,b]$ and every $y \in Y$ the inequality

$$|L(f,y) - A(f,y)| \leq (\|L\| + \|A\|)\cdot\Omega\left(f; \frac{\Phi(y)}{\|L\|+\|A\|}, \frac{\gamma_1(Y)}{\Phi(Y)}, \frac{\gamma_2(Y)}{\Phi(Y)}\right)$$

holds.

Proof. For $f \in C[a,b]$, $f_1 \in C^1[a,b]$, and $f_2 \in C^2[a,b]$ by our assumptions the following estimate holds:

$$\begin{aligned}|(L-A)(f,y)| &= |(L-A)(f - f_1 + f_1 - f_2 + f_2, y)| \\ &\leq |(L-A)(f-f_1,y)| + |(L-A)(f_1-f_2,y)| + |(L-A)(f_2,y)| \\ &\leq (\|L\|+\|A\|)\cdot\|f-f_1\| + \Phi(y)\cdot\|f_1'-f_2'\| + \gamma_1(y)\cdot\|f_2'\| + \\ &\quad + \gamma_2(y)\cdot\|f_2''\|.\end{aligned}$$

In other words,

$$|(L-A)(f,y)| \leq (\|L\|+\|A\|)\cdot\|f-f_1\| + \Phi(y)\left\{\|f_1'-f_2'\| + \frac{\gamma_1(y)}{\Phi(y)}\cdot\|f_2'\| + \frac{\gamma_2(y)}{\Phi(y)}\cdot\|f_2''\|\right\}.$$

Passing to the infimum over $f_2 \in C^2[a,b]$ implies

$$|(L-A)(f,y)| \leq (\|L\|+\|A\|)\cdot\|f-f_1\| + \Phi(y)\cdot\tilde{K}\left(\frac{\gamma_1(y)}{\Phi(y)}, \frac{\gamma_2(y)}{\Phi(y)}, f_1; C^1, C^2\right).$$

This can be rewritten in the form

$$|(L-A)(f,y)| \leq (\|L\|+\|A\|)\cdot\left\{\|f-f_1\| + \frac{\Phi(y)}{\|L\|+\|A\|}\cdot\tilde{K}\left(\frac{\gamma_1(y)}{\Phi(y)}, \frac{\gamma_2(y)}{\Phi(y)}, f_1; C^1, C^2\right)\right\}.$$

Passing again to the infimum over $f_1 \in C^1[a,b]$ we obtain

$$\begin{aligned}|(L-A)(f,y)| &\leq (\|L\|+\|A\|)\cdot K\left(\frac{\Phi(y)}{\|L\|+\|A\|}, f; (C, \|\cdot\|), \left(C^1, \tilde{K}\left(\frac{\gamma_1(y)}{\Phi(y)}, \frac{\gamma_2(y)}{\Phi(y)}, *\right)\right)\right) = \\ &= (\|L\|+\|A\|)\cdot\Omega\left(f; \frac{\Phi(y)}{\|L\|+\|A\|}, \frac{\gamma_1(y)}{\Phi(y)}, \frac{\gamma_2(y)}{\Phi(y)}\right). \quad \blacksquare\end{aligned}$$

Our next theorem treats the case where L is a positive linear operator satisfying the additional assumption that $L(e_0,y) = A(e_0,y)$. If L does not have this property this difficulty can be circumvented by using Proposition 4.2.

THEOREM 4.4 *Let $L: C[a,b] \to B(Y)$ be a positive linear operator and let $A \neq 0$ be given as above. If $L(e_0,y) = A(e_0,y)$ for all $y \in Y$, the for all $f \in C[a,b]$ and all $y \in Y$ the estimate*

$$|(L-A)(f,y)| \leq 2\cdot\|L\|\cdot\Omega\left(f; \frac{L(|e_1 - g_A(y)|, y)}{2\cdot\|L\|}, \frac{|L(e_1 - g_A(y), y)|}{L(|e_1 - g_A(y)|, y)}, \frac{L((e_1 - g_A(y))^2, y)}{2L(|e_1 - g_A(y)|, y)}\right)$$

holds. The estimate is also true if one or more of the three 'differences' occurring on the right side are replaced by majorants, such that the appearing quotients remain finite.

Proof. In order to prove the above inequality we have to find suitable functions Φ, γ_1, and γ_2 in Theorem 4.3. If A is given by $A(f,y) = A(e_0,y)\cdot f(g_A,y) = L(e_0,y)\cdot f(g_A,y)$, then we have for every $f_1 \in C^1[a,b]$ and every $y \in Y$:

$$\begin{aligned}|L(f_1,y) - A(f_1,y)| &= |L(f_1 - f_1(g_A(y)), y)| \\ &\leq L(|e_1 - g_A(y)|, y)\cdot\|f_1'\| \\ &=: \Phi(y)\cdot\|f_1'\|.\end{aligned}$$

If $f_2 \in C^2[a,b]$, we proceed as follows: We interpolate f_2 at the point $g_A(y)$ by a polynomial p of first degree satisfying the properties

$$p(g_A(y)) = f_2(g_A(y)) \text{ and } p'(g_A(y)) = f_2'(g_A(y)).$$

This polynomial is given by

$$p(t) = f_2(g_A(y)) + f_2'(g_A(y))\cdot(t - g_A(y)).$$

Then it is known that

$$|f_2(t) - p(t)| = |f_2(t) - [f_2(g_A(y)) + f_2'(g_A(y))\cdot(t - g_A(y))]| = \left|f_2''(\xi)\cdot\frac{(t-g_A(y))^2}{2}\right| \le \frac{1}{2}\cdot\|f_2''\|\cdot(t-g_A(y))^2,$$

where $\xi \in [\min\{t, g_A(y)\}, \max\{t, g_A(y)\}]$. We now consider the difference $(L - A)(f_2)$. First of all we write

$$|L(f_2, y) - A(f_2, y)| = |L(f_2, y) - L(p, y) + L(p, y) - A(f_2, y)|.$$

Here,

$$A(f_2, y) = A(e_0, y)\cdot f_2(g_A(y)) + f_2'(g_A(y))\cdot[A(e_1, y) - g_A(y)\cdot A(e_0, y)] = A(p, y).$$

Thus,

$$|L(f_2, y) - A(f_2, y)| = |L(f_2, y) - L(p, y) + L(p, y) - A(p, y)| \le L(|f_2 - p|, y) + |L(p, y) - A(p, y)| \le \frac{1}{2}\cdot L((e_1 - g_A(y))^2, y)\cdot\|f_2''\| + |L(p, y) - A(p, y)|.$$

It remains to investigate the second term on the right-hand side of this inequality. We have

$$|(L - A)(p, y)| = |f_2'(g_A(y))|\cdot|(L - A)(e_1, y)| \le \|f_2'\|\cdot|(L - A)(e_1, y)| = \|f_2'\|\cdot|L(e_1 - g_A(y), y)|,$$ which follows from $L(e_0, y) = A(e_0, y)$.

Putting now $\gamma_1(y) := |L(e_1 - g_A(y), y)|$ and $\gamma_2(y) := \frac{1}{2}\cdot L((e_1 - g_A(y))^2, y)$, and observing the fact that $\|L\| = \|A\|$ we arrive at the inequality in Theorem 4.4. ▮

The following corollary of Theorem 4.4 is of particular importance. It deals with the approximation of the identity by positive linear operators.

Corollary 4.5 *If Y is a non-empty subset of an interval $[a, b]$ and if $L: C[a, b] \to B(Y)$ is a positive linear operator satisfying $L(e_0, \cdot) = e_0$, then for any $f \in C[a, b]$, and all $x \in Y$ the estimate*

$$|L(f, x) - f(x)| \le 2\cdot\Omega\left(f; \frac{L(|e_1 - x|, x)}{2}, \frac{|L(e_1 - x, x)|}{|L(e_1 - x|, x)}, \frac{L((e_1 - x)^2, x)}{2\cdot L(|e_1 - x|, x)}\right)$$

holds. The estimate remains true if one or more of the three 'differences' occurring on the right side are replaced by majorants such that the quotients remain finite.

Another consequence of Theorem 4.4 is a Korovkin type theorem for the approximation of almost lattice homomorphisms A mapping $C[a, b]$ into $B(Y)$. This will be seen in a corollary of the following Theorem 4.6, where we show as an example how Theorem 4.4 and the estimate given for Ω in Theorem 3.5 can be combined.

Theorem 4.6 *Let Y be a non-empty set and $[a, b]$ a compact interval of the real axis. If $A: C[a, b] \to B(Y)$ is an almost lattice homomorphism given by $A(f, y) = A(e_0, x)\cdot f(g_A(y))$ and if L is a positive linear operator between the above spaces, then for all $f \in C[a, b]$, all $y \in Y$ and each $h > 0$ the following estimate holds:*

$$|L(f, y) - A(f, y)| \le \left[3\cdot\|L\| + \max\left(\frac{1}{h^2}, \frac{1}{(b-a)^2}\right)\cdot L((e_1 - g_A(y))^2, y)\right]\cdot\omega_2(f, h) + \left[2\cdot\max\left(\frac{1}{h}, \frac{1}{b-a}\right)\cdot|L(e_1 - g_A(y), y)|\right]\cdot\omega_1(f, h) + |L(e_0, y) - A(e_0, y)|\cdot\|f\|.$$

Proof. In view of Proposition 4.2 we first have

$$|L(f, y) - A(f, y)| \le |L(f, y) - L(e_0, y)\cdot f(g_A(y))| + |L(e_0, y) - A(e_0, y)|\cdot\|f\|.$$

Now $\tilde{A}$ given by $\tilde{A}(f, y) = L(e_0, y)\cdot f(g_A(y))$ is a lattice homomorphism satisfying $\tilde{A}(e_0, y) = L(e_0, y)$.

Theorem 4.4 yields the inequality

$$(L - \tilde{A})(f, y)| \le 2\cdot\|L\|\cdot\Omega(f; t, t_1, t_2),$$

where $t = \dfrac{L(|e_1 - g_A(y)|, y)}{2\cdot\|L\|}$, $t_1 = \dfrac{|L(e_1 - g_A(y), y)|}{L(|e_1 - g_A(y)|, y)}$,

$$t_2 = \frac{L((e_1 - g_A(y))^2, y)}{2\cdot L(|e_1 - g_A(y)|, y)}.$$

Theorem 3.5 now implies

$$|(L - \tilde{A})(f, y)| \le \left[3\cdot\|L\| + L((e_1 + g_A(y))^2, y)\cdot\max\left(\frac{1}{h^2}, \frac{1}{(b-a)^2}\right)\right]\cdot\omega_2(f, h) + \left[2\cdot|L(e_1 - g_A(y), y)|\cdot\max\left(\frac{1}{h}, \frac{1}{b-a}\right)\right]\cdot\omega_1(f, h).$$

Together with the observation made at the beginning of the proof we obtain the desired inequality. ▮

The following corollary which we state without giving a proof shows that Theorem 4.6 indeed contains a Korovkin type theorem for the approximation of almost lattice homomorphisms.

Corollary 4.7 *Let* $Y, [a, b]$, *and* A *be given as in Theorem* 4.6. *Suppose that* $Y_0 \neq \emptyset$ *is as subset of* Y. *For* $f \in B(Y)$ *we put* $\|f\|_{Y_0} := \sup\{|f(y)| : y \in Y_0\}$. *If* $L_n : C[a, b] \to B(Y)$, $n \geq 1$, *is a sequence of positive linear operators, then the following are equivalent :*

(*i*) $\quad \| L_n f - Af \|_{Y_0} \to 0$ *for all* $f \in C[a, b]$, $n \to \infty$,

(*ii*) $\quad \| L_n e_i - A e_i \|_{Y_0} \to 0$ *for* $i \in \{0, 1, 2\}$, $n \to \infty$.

V. Examples: Refined Estimates for Approximation by Special Linear Operators. After what has been said before it is clear that the combination of Theorem 4.4 or Corollary 4.5 with the results obtained in Section III yields a variety of propositions concerning the approximation by certain linear operators. It is the aim of this section to show that many of these combinations give estimates which are better than the ones known until now. We restrict ourselves to three prominent examples : Bernstein polynomials, Meyer-König and Zeller operators, and Hermite-Fejér type interpolation operators.

1. *Pointwise Approximation by Bernstein Polynomials.* In this subsection we discuss what our theorems yield for the approximation by Bernstein operators given by the formula

$$B_n : C[0, 1] \to \Pi_n[0, 1],\ B_n(f, x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1 - x)^{n-k}.$$

It is well known that $B_n(e_i, x) = x^i (i = 0, 1)$ for all $x \in [0, 1]$ and all $n \geq 1$. Thus the quantities appearing in Corollary 4.5 can be written as follows

$$|B_n(e_1 - x, x)| = 0,\ B_n((e_1 - x)^2, x) = \frac{x(1-x)}{n}, \text{ and}$$

$$B_n(|e_1 - x|, x) = \frac{2}{n}(n - r)\binom{n}{r} x^{r+1}(1 - x)^{n-r},$$

where r is given by $r = [nx]$ and where $[nx]$ denotes the largest integer not exceeding nx (see F. Schurer and F.W. Steutel [15]). However, we shall frequently use the estimate $B_n(|e_1 - x|, x) \leq (B_n((e_1 - x)^2, x))^{1/2}$.

We assume first that f is in $C^1[0,1]$. For this case, using $\omega_1(f', \cdot)$, very good estimates were given by F. Schurer and F. W. Steutel [15]. A certain analogue using the least concave majorant of $\omega_1(f', \cdot)$ is given in

Theorem 5.1. *For any* $f \in C^1[0, 1]$, $x \in [0, 1]$, $n \geq 1$ *and* $h > 0$ *we have*

$$|B_n(f, x) - f(x)| \leq \frac{1}{2}\cdot\max\left(B_n(|e_1 - x|, x), \frac{1}{h}\cdot B_n((e_1 - x)^2, x)\right)\cdot\tilde{\omega}_1(f', h)$$

$$\leq \frac{1}{2}\max\left(\left(\frac{x(1-x)}{n}\right)^{1/2}, \frac{x(1-x)}{nh}\right)\cdot\tilde{\omega}_1(f', h).$$

The *proof* is obtained by combining Corollaries 3.2 and 4.5.

Corollary 5.2 (*i*) *For the particular choice* $h = \left(\frac{x(1-x)}{n}\right)^{1/2}$ *the above inequality implies*

$$|B_n(f, x) - f(x)| \leq \frac{1}{2}\cdot\left(\frac{x(1-x)}{n}\right)^{1/2}\cdot\tilde{\omega}_1\left(f', \left(\frac{x(1-x)}{n}\right)^{1/2}\right)$$

$$\leq \left(\frac{x(1-x)}{n}\right)^{1/2}\cdot\omega_1\left(f', \left(\frac{x(1-x)}{n}\right)^{1/2}\right).$$

(*ii*) *We also have the uniform Lorentz type estimates*

$$\| B_n f - f \| \leq \frac{1}{4}\cdot n^{-1/2}\cdot\tilde{\omega}_1\left(f', \frac{1}{2}\cdot n^{-1/2}\right) \leq \frac{1}{2}\cdot n^{-1/2}\cdot\omega_1\left(f', \frac{1}{2}\cdot n^{-1/2}\right)$$

Remarks. (*i*) If x and n are fixed, for $h \leq \left(\frac{x(1-x)}{n}\right)^{1/2}$ the function $c_n(x, h) := \frac{1}{2}\cdot\max\left(\left(\frac{x(1-x)}{n}\right)^{1/2}, \frac{x(1-x)}{nh}\right)$ from Theorem 5.1 is best possible in the sense that the estimate becomes an equality for $f_x(t) := (t - x)^2$.
(*ii*) A similar statement is true concerning the first inequality in Corollary 5.2 (*ii*) : the constant $1/4$ cannot be replaced by any number $c < 1/4$.
(*iii*) The second estimate in Corollary 5.2 (*ii*) should be compared to another result by Schurer and Steutel who showed that $\|B_n f - f\| \leq 1/4\cdot n^{-1/2}\cdot\omega_1(f', n^{-1/2})$. As can be seen from Corollary 5.2, for functions f with $\omega_1(f', \cdot)$ concave we even obtain

$$\|B_n f - f\| \leq 1/4\cdot n^{-1/2}\cdot\omega_1(f', 1/2\cdot n^{-1/2}).$$

Our next theorem deals with approximation of arbitrary continuous functions.

Theorem 5.3. *For* $f \in C[0, 1]$, $x \in [0, 1]$, $n \geq$ *and* $h > 0$ *we have :*

(*i*) $$|B_n(f, x) - f(x)| \leq \max\left(1, \frac{B_n(|e_1 - x|, x)}{h}\right)\cdot\tilde{\omega}_1(f, h),$$

(*ii*) $$|B_n(f, x) - f(x)| \leq 2\cdot\max\left(1, \frac{B_n(|e_1 - x|, x)}{h}\right)\cdot\omega_1^*(f, h).$$

Proof. Inequality (*i*) is obtained by combining Corollary 4.5 and Theorem 3.3 ; the second estimate is due to the fact that B_n reproduces functions. ∎

Corollary 5.4 *Under the assumptions of Theorem 5.3 one gets for* $h = \left(\frac{x(1-x)}{n}\right)^{1/2}$

(*i*) $$|B_n(f, x) - f(x)| \leq \tilde{\omega}_1\left(f, \left(\frac{x(1-x)}{n}\right)^{1/2}\right) \leq \tilde{\omega}_1(f, 1/2\cdot n^{-1/2}).$$

(ii) $$|B_n(f, x) - f(x)| \leq 2 \cdot \omega_1^*\left(f, \left(\frac{x(1-x)}{n}\right)^{1/2}\right) \leq 2 \cdot \omega_1^*(f, 1/2 \cdot n^{-1/2}).$$

Remark. As can be seen from the choices $n = 1$ and $f_0 = \left|e_1 - \frac{1}{2}\right|$ in both estimates in Corollary 5.4 (i) the constant 1 in front of $\tilde{\omega}_1(f, \cdot)$ cannot be replaced by any number $c < 1$.

Our last example concerning Bernstein operators arises from the combination of Corollaries 4.5 and 3.6.

THEOREM 5.5 *For any* $f \in C[0, 1]$, $x \in [0, 1]$, $n \geq 1$, *and* $h > 0$ *the following inequality holds :*

$$|B_n(f, x) - f(x)| \leq \left[3 + \frac{x(1-x)}{n} \cdot \max\left(\frac{1}{h^2}, 1\right)\right] \cdot \omega_2(f, h).$$

COROLLARY 5.6 *For* $h = \left(\frac{x(1-x)}{2n}\right)^{1/2}$, $x \in (0, 1)$, *the above inequality leads to*

$$|B_n(f, x) - f(x)| \leq 5 \cdot \omega_2\left(f, \left(\frac{x(1-x)}{2n}\right)^{1/2}\right) \textit{for all } x \in [0, 1].$$

As far as the constant in front of $\omega_2(f, \cdot)$ is concerned this is an improvement of an inequality given by H. Berens and G. G. Lorentz [2]. For $h = \left(\frac{x(1-x)}{n}\right)^{1/2}$ we get

$$|B_n(f, x) - f(x)| \leq 4 \cdot \omega_2\left(f, \left(\frac{x(1-x)}{n}\right)^{1/2}\right)$$

which refines a recent estimate of L. I. Strukov and A. F. Timan [19,20]. The choice $h = \left(\frac{4x(1-x)}{n}\right)^{1/2}$ yields

$$|B_n(f, x) - f(x)| \leq 3.25 \cdot \omega_2\left(f, \left(\frac{4x(1-x)}{n}\right)^{1/2}\right).$$

This is a pointwise version of a result first proved by Yu. A. Brudnyĭ [3]. It is not yet known what the optimal constants in the above inequalities are.

After this look at the Bernstein operators it should have become clear that it is worthwhile to combine the theorems in Section IV with the estimates for Ω as given in Section III in order to arrive at a variety of refined inequalities sometimes even giving optimal constants. It has to be mentioned that there are a number of modifications of the Bernstein operators to which our above theorems can be applied. For these the reader is referred for instance to two bibliographies which have recently been compiled [17, 18, 9].

2. *Pointwise Approximation by the Operators of Meyer-König and Zeller.* This subsection parallels the one for Bernstein polynomials. The operators of Meyer-König and Zeller are given by the formula ($n \in \mathbb{N}$, $x \in [0, 1]$, $f \in C[0, 1]$)

$$M_n(f, x) := (1-x)^{n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k f\left(\frac{k}{n+k}\right), \text{ if } 0 \leq x < 1,$$

and

$$M_n(f, 1) := f(1).$$

These operators are also positive and linear and satisfy the equations $M_n e_i = e_i$ for $i = 0, 1$. Thus we have $M_n(e_1 - x, x) = 0$ for $0 \leq x \leq 1$. An explicit representation for $M_n((e_1 - x)^2, x)$ was only recently given by J.A.H. Alkemade [1] who proved

$$M_n((e_1 - x)^2, x) = \frac{x(1-x)^2}{n+1} \cdot {}_2F_1(1, 2; n+2; x) \leq \frac{4}{27n+9}$$

for $0 \leq x \leq 1$ and $n \geq 2$. Here ${}_2F_1(1, 2; n+2; x)$ denotes the hypergeometric series given by

$$\sum_{k=0}^{\infty} \frac{(1)_k (2)_k}{(n+2)_k} \cdot \frac{x^k}{k!}.$$

An explicit representation of the quantity $M_n(|e_1 - x|, x)$ was derived by F. Schurer and F. W. Steutel [16]. However, as in Section V.1 we shall use the estimate $M_n(|e_1 - x|, x) \leq (M_n((e_1 - x)^2, x))^{1/2}$ which simplifies some inequalities. Again we investigate differentiable functions first.

THEOREM 5.7. *For any* $f \in C^1[0, 1]$, $x \in [0, 1]$, $n \geq 2$ *and* $h > 0$ *the following inequalities hold :*

(i) $$|M_n(f, x) - f(x)| \leq$$
$$\leq \frac{1}{2} \cdot \max\left[M_n(|e_1 - x|, x), \frac{1}{h} \cdot M_n((e_1 - x)^2, x)\right] \cdot \tilde{\omega}_1(f', h),$$

(ii) $$|M_n(f, x) - f(x)| \leq$$
$$\leq \left[M_n(|e_1 - x|, x) + \frac{M_n((e_1 - x)^2, x)}{2h}\right] \cdot \omega_1(f', h).$$

Proof. Both statements are obtained by combining Corollaries 3.2 and 4.5. ∎

COROLLARY 5.8 *Under the assumptions of Theorem 5.7 we also have :*

(i) $$|M_n(f, x) - f(x)| \leq \frac{1}{2}\left(\frac{x(1-x)^2}{n+1} {}_2F_1(1,2; n+2; x)\right)^{1/2} \cdot$$
$$\cdot \tilde{\omega}_1\left(f', \left(\frac{x(1-x)^2}{n+1} \cdot {}_2F_1(1, 2; n+2; x)\right)^{1/2}\right),$$

(ii) $$|M_n(f,x)-f(x)|\le\frac{1}{2}\left(\frac{x(1-x)^2}{n-1}\right)^{1/2}\cdot\tilde{\omega}_1\left(f',\left(\frac{x(1-x)^2}{n-1}\right)^{1/2}\right),$$

(iii) $$\|M_nf-f\|\le\frac{1}{3\sqrt{3n+1}}\cdot\tilde{\omega}_1\left(f',\frac{2}{3\sqrt{3n+1}}\right)\le$$

$$\le\frac{2}{3\sqrt{3n+1}}\cdot\omega_1\left(f',\frac{2}{3\sqrt{3n+1}}\right).$$

Proof. (*i*) is obtained by using the estimate $M_n(|e_1-x|,x)\le$ $\le(M_n((e_1-x)^2,x))^{1/2}$ and substituting $h=(M_n((e_1-x)^2,x))^{1/2}$. For (*ii*) we use the inequality ${}_2F_1(1,2;n+2;x)\le{}_2F_1(1,2;n+2;1)=$ $=\frac{n+1}{n-1}$, $n\ge2$.

Thus we get

$$|M_n(f,x)-f(x)|\le\frac{1}{2}\left(\frac{x(1-x)^2}{n-1}\right)^{1/2}\cdot\tilde{\omega}_1\left(f',\left(\frac{x(1-x)^2}{n-1}\right)^{1/2}\right)$$

For the proof of (*iii*) observe that $\frac{x(1-x)^2}{n+1}\cdot{}_2F_1(1,2;n+2;x)\le$ $\le\frac{4}{27n+9}$. The inequality from (*i*) implies

$$\|M_nf-f\|\le\frac{1}{3\sqrt{3n+1}}\cdot\tilde{\omega}_1\left(f',\frac{2}{3\sqrt{3n+1}}\right)\le$$

$$\le\frac{2}{3\sqrt{3n+1}}\cdot\omega_1\left(f',\frac{2}{3\sqrt{3n+1}}\right).\ \blacksquare$$

Remark. (*i*) As was the case for Bernstein operators, the constant $\frac{1}{2}$ in Corollary 5.8 (i) cannot be replaced by any number $c<\frac{1}{2}$.
(*ii*) If f' has a concave modulus of continuity, then the first inequality in 5.8 (*iii*) even reads

$$\|M_nf-f\|\le\frac{1}{3\sqrt{3n+1}}\cdot\omega_1\left(f',\frac{2}{3\sqrt{3n+1}}\right).$$

For any function of the form $f(t)=at^2+bt+c$ it becomes an equality and thus the above estimate is best possible in a certain sense. The second estimate in 5.8 (*iii*) is an improvement of Theorem 8 in [1].
(*iii*) It would also have been possible to use the second estimate in Theorem 5.7 to obtain e.g. improvements of Alkemade's result. However, in most cases the first inequality gives better majorants.

For the operators M_n the analogue of Theorem 5.3 is given in

THEOREM 5.9 *For $f\in C[0,1]$, $x\in[0,1]$, $n\ge2$ and any $h>0$ there holds*

$$|M_n(f,x)-f(x)|\le\max\left(1,\frac{M_n(|e_1-x|,x)}{h}\right)\cdot\tilde{\omega}_1(f,h).$$

COROLLARY 5.10. *Under the assumptions of Theorem 5.9 the following inequalities are true :*

(i) $$|M_n(f,x)-f(x)|\le\tilde{\omega}_1\left(f,\left(\frac{x(1-x)^2}{n+1}\cdot{}_2F_1(1,2;n+2;x)\right)^{1/2}\right),$$

(ii) $$|M_n(f,x)-f(x)|\le\tilde{\omega}_1\left(f,\left(\frac{x(1-x)^2}{n-1}\right)^{1/2}\right),$$

(iii) $$|M_n(f,x)-f(x)|\le2\cdot\omega_1^*\left(f,\left(\frac{x(1-x)^2}{n-1}\right)^{1/2}\right).$$

Proof. The above inequalities are obtained by using again the Cauchy-Schwarz inequality, the explicit representation of $M_n((e_1-x)^2,x)$ as given by Alkemade, the inequalities $\tilde{\omega}_1(f,\cdot)\le2\cdot\omega_1(f,\cdot)$ and ${}_2F_1(1,2;$ $n+2;x)\le(n+1)/(n-1)$ for $0\le x\le1$, $n\ge2$, and the fact that M_n reproduces linear functions.

We now turn to estimates involving the second order modulus of smoothness. For the operators M_n one has

THEOREM 5.11 *For any $f\in C[0,1]$, $x\in[0,1]$, $n\ge2$, and $h>0$ there holds*

$$|M_n(f,x)-f(x)|\le\left[3+\frac{x(1-x)^2}{n+1}\cdot{}_2F_1(1,2;n+2;x)\cdot\right.$$

$$\left.\cdot\max\left(\frac{1}{h^2},1\right)\right]\cdot\omega_2(f,h).$$

COROLLARY 5.12 *Under the assumptions of Theorem 5.11 the following inequalities are true :*

(i) $$|M_n(f,x)-f(x)|\le\left(3+\frac{n+1}{n-1}\right)\cdot\omega_2\left(f,\left(\frac{x(1-x)^2}{n+1}\right)^{1/2}\right),$$

(ii) $$\|M_nf-f\|\le3\frac{4}{21}\cdot\omega_2(f,(n+1)^{-1/2}).$$

Proof. (*i*) is a simple consequence of the substitution $h=$ $=\left(\frac{x(1-x)^2}{n+1}\right)^{1/2}$ and the fact that ${}_2F_1(1,2;n+2;x)\le\frac{n+1}{n-1}$. For the proof of (*ii*) we use the uniform estimate $\|M_ne_2-e_2\|\le\frac{4}{27n+9}$.

Substituting $h = (n+1)^{-1/2}$ into the inequality in Theorem 5.11 yields

$$|M_n(f, x) - f(x)| \le \left[3 + \frac{4}{27n+9}\cdot(n+1)\right]\cdot\omega_2(f, (n+1)^{-1/2})$$

$$\le 3\frac{4}{21}\cdot\omega_2(f, (n+1)^{-1/2}). \blacksquare$$

Both inequalities in Corollary 5.12 improve as obtained by the author earlier (see [6]). However, to our knowledge the exact value of e.g.

$$\sup_{n\ge 2}\ \sup_{\substack{f\in C[0,1]\\ f\neq \text{linear}}} \frac{\|M_n f - f\|}{\omega_2(f, (n+1)^{-1/2})}$$

has not yet been determined. A similar statement holds for Bernstein operators.

3. *Pointwise Approximation by Hermite-Fejér Type Interpolation Polynomials.* This subsection is devoted to pointwise approximation by Hermite-Fejér interpolation polynomials which have been investigated intensively. For a good survey see e.g. the 'Habilitationsschrift' of H. B. Knoop [10] or the bibliography in a paper of T.Y. Mills [12]. As was shown in [8], it is possible to prove pointwise estimates for the approximation of functions in $C^1[-1,1]$ which are of order $0(1/n)$ for $f' \in \text{Lip } 1$ and at the same time reproduce the interpolation conditions. However, these estimates are rather lengthy. This is why we shall investigate slightly modified operators which satisfy simpler estimates of the same order. This will show at the same time that the general theory developed so far is also applicable to the case of non-positive linear operators. For the sake of brevity we shall only consider the most prominent example.

The classical Hermite-Fejér polynomials $H_n(f, \cdot)$ interpolating a function $f \in \mathbb{R}^{[-1,1]}$ at the zeros of the Čebyšev polynomials T_n and having a derivative equal to zero at these points, i.e.

$$H_n(f, x) = \sum_{k=1}^{n} f(x_k)\cdot\frac{(1 - x x_k)T_n(x)^2}{n^2(x - x_k)^2},$$

where $x_k = \cos((2k-1)\pi/2n)$, satisfy, for $f \in C^1[-1,1]$, $x \in [-1,1]$ and $n \ge 2$, the following inequality (see [8])

$$|H_n(f, x) - f(x)|$$

$$\le \frac{1}{n}|T_n(x)\cdot T_{n-1}(x)|\cdot\|f'\| + \frac{c_1}{n}|T_n(x)|\cdot(1 + \sqrt{1-x^2}\log n)\cdot$$

$$\cdot\,\omega_1\left(f', \left(\frac{|T_n(x)|}{1+\sqrt{1-x^2}\cdot\log n}\right)\right).$$

Unfortunately the term containing $\|f'\|$ cannot be removed from this estimate which is due to the fact that H_n does not reproduce straight lines except for constants. However, it is possible to get rid of this quantity

by passing from the positive operator H_n to a non-positive linear operator $\bar{H}_n$ yielding appropriate slopes at the x_k's. In order to be somewhat more general we prove the following theorem which describes the background of our following considerations for the classical Hermite-Fejér operators. It is of advantage for many other interpolation schemes as well.

THEOREM 5.13 *If $Q: C[-1,1] \to C[-1,1]$ is a positive linear operator with $Qe_0 = e_0$ and if for $f \in C[-1,1]$ the expression Lf denotes the linear function interpolating f at -1 and 1, then for the operator $\bar{Q}$ given by $\bar{Q}(f, x) := Q(f - Lf, x) + Lf(x)$ the following inequalites hold:*

(i) $\|\bar{Q}\| \le 3,$

(ii) $|\bar{Q}(g, x) - g(x)| \le 2\cdot Q(|e_1 - x|, x)\cdot\|g'\|$ *for all* $g \in C^1[-1,1]$,

(iii) $|\bar{Q}(h, x) - h(x)| \le \left[\frac{1}{2}\cdot Q((e_1 - x)^2, x) + 2\cdot|Q(e_1 - x, x)|\right]\cdot\|h''\|$

for all $h \in C^2[-1,1]$.

Proof. (i) follows from $\|\bar{Q}f\| \le \|Q(f - Lf)\| + \|Lf\| \le 3\cdot\|f\|$.
(ii) can be obtained as follows. For $g \in C^1[-1,1]$ we have

$$|\bar{Q}(g, x) - g(x)| = |Q(g - Lg, x) - (g - Lg)(x)| \le Q(|e_1 - x|, x)\cdot\|(g - Lg)'\|$$

$$\le 2\cdot Q(|e_1 - x|, x)\cdot\|g'\|.$$

Here the first inequality follows from the fact that Q is positive and satisfies the equation $Qe_0 = e_0$.
(iii) Proceeding as in the proof of Theorem 4.4 yields the inequality

$$|\bar{Q}(h, x) - h(x)| = |Q(h - Lh, x) - (h - Lh)(x)|$$

$$\le \frac{1}{2}\cdot Q((e_1 - x)^2, x)\cdot\|(h - Lh)''\| + |Q(e_1 - x, x)|\cdot\|(h - Lh)'\|$$

$$\le \left[\frac{1}{2}\cdot Q((e_1 - x)^2, x) + 2\cdot|Q(e_1 - x, x)|\right]\cdot\|h''\|. \blacksquare$$

As can be seen from the formulation of Theorem 5.13 it prepares the operators $\bar{Q}_n$ for an application of the above Theorem 4.3 and a combination with suitable results from Section III.

Remark. The method used in Theorem 5.13, namely the perturbing of a positive linear operator in order to obtain simpler estimates, is only one particular example of a more general technique. This technique, which will be described elsewhere, allows to impose side conditions of different types on a given sequence of linear operators.

Using the above theorem it is now easy to give shorter estimates for a version of the classical Hermite-Fejér operators which reproduce straight lines. We restrict ourselves to the case of continuously differentiable functions.

THEOREM 5.14 *Let $H_n: C[-1,1] \to C[-1,1]$ denote the classical Hermite-Fejér operator described above. If L is given as in Theorem 5.13,* *then the perturbed operator* $\overline{H}_n$ *given by the formula* $\overline{H}_n f := H_n(f - Lf) + Lf$ *satisfies for* $f \in C^1[-1,1]$, $x \in [-1,1]$, $n \geq 2$, *the inequality*

$$|\overline{H}_n(f, x) - f(x)| \leq \frac{5c_1}{n} \cdot |T_n(x)| \cdot (1 + \sqrt{1-x^2}\log n) \cdot \omega_1\left(f', \frac{1}{1+\sqrt{1-x^2}\log n}\right).$$

Here c_1 *is the optimal constant in* $H_n(|e_1 - x|, x) \leq \frac{c_1}{n} |T_n(x)| \cdot (1 + \sqrt{1-x^2}\log n)$.

Proof. We use the (in)equalities (cf. [8]).

$$H_n(|e_1 - x|, x) \leq \frac{c_1}{n} |T_n(x)| \cdot (1 + \sqrt{1-x^2}\log n),$$

$$H_n((e_1 - x)^2, x) = \frac{1}{n} \cdot T_n^2(x), \text{ and } H_n(e_1 - x, x) = -\frac{1}{n} \cdot T_n(x) \cdot T_{n-1}(x).$$

From the last equality it follows that $H_n(|e_1 + 1|, -1) = H_n(e_1 + 1, -1) = \frac{1}{n}$.

Thus, $c_1 \geq 1$.

Theorem 5.13 yields the estimates:

(*i*) $\quad \|\overline{H}_n\| \leq 3.$

(*ii*) $\quad |\overline{H}_n(g, x) - g(x)| \leq 2\frac{c_1}{n} |T_n(x)| (1 + \sqrt{1-x^2}\log n) \cdot \|g'\|$

for $g \in C^1[-1, 1]$,

(*iii*) $\quad |\overline{H}_n(h, x) - h(x)| \leq \left[\frac{1}{2n} \cdot T_n^2(x) + \frac{2}{n} |T_n(x) \cdot T_{n-1}(x)|\right] \cdot \|h''\|$

for $h \in C^2[-1, 1]$.

Theorem 4.3 now tells us that for an arbitrary $f \in C^1[-1, 1]$ the inequality

$$|\overline{H}_n(f, x) - f(x)| \leq 4 \cdot \Omega\left(f; \frac{c_1 \cdot |T_n(x)|(1 + \sqrt{1-x^2}\log n)}{2n}, 0, \frac{|T_n(x)| + 4|T_{n-1}(x)|}{4c_1 \cdot (1 + \sqrt{1-x^2}\log n)}\right)$$

is true, and Corollary 3.2 shows that the latter quantity can be estimated by

$$2 \cdot \frac{c_1 \cdot |T_n(x)|(1 + \sqrt{1-x^2}\log n)}{n} \left(1 + \frac{|T_n(x)| + 4|T_{n-1}(x)|}{4c_1 \cdot (1 + \sqrt{1-x^2}\log n) \cdot h}\right) \cdot \omega_1(f', h),$$

where $h > 0$ is a given positive number. Choosing $h = 1/(1 + \sqrt{1-x^2}\log n)$ yields the desired inequality. ∎

Remark. With respect to the constant c_1 appearing in our last theorem we would like to mention the following: Some numerical evidence suggests that it equals 1. As far as we know this has not been proven yet. It would imply, among others, the following estimate for the classical Hermite-Fejér operators

$$|H_n(f, x) - f(x)| \leq \tilde{\omega}_1\left[f, \frac{1}{n} \cdot |T_n(x)| \cdot (1 + \sqrt{1-x^2}\log n)\right].$$

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