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# PROBABILISTIC APPROACH TO A CLASS OF GENERALIZED BERNSTEIN APPROXIMATING OPERATORS

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In our previous paper [13] we have introduced a sequence of positive linear operators which can be useful in the constructive approximation theory : $(L_{m,r}^{\langle\alpha\rangle})$, where $r$ is a non-negative integer parameter, $m$ is a natural number such that $m > 2r$, while $\alpha$ is a non-negative parameter which may depend on $m$. To each function $f : [0,1] \to \mathbb{R}$ we have associated the operator $L_{m,r}^{\langle\alpha\rangle}$, defined by

$$(1)\qquad (L_{m,r}^{\langle\alpha\rangle} f)(x) := \sum_{k=0}^{m-r} p_{m-r,k}^{\langle\alpha\rangle}(x)\left\{[1 - x + (m-r-k)\alpha]\, f\left(\frac{k}{m}\right) + (x + k\alpha)\, f\left(\frac{k+r}{m}\right)\right\},$$

where, in terms of factorial powers :

$$t^{[n,h]} := t(t-h)\ldots(t-(n-1)h),\quad t^{[0,h]} := 1,$$

we have

$$(2)\qquad p_{m-r,k}^{\langle\alpha\rangle}(x) := \binom{m-r}{k}\frac{x^{[k,-\alpha]}(1-x)^{[m-r-k,-\alpha]}}{(1+\alpha)^{[m-r,-\alpha]}}.$$

It should be noticed that the operator corresponding to the case $\alpha = 0$ has been investigated in detail in our earlier paper [12].

Now we shall indicate a probabilistic way to arrive at these operators and will present some basic approximation properties of these operators.

For simplicity, we shall imagine an urn model which leads us to a discrete probability distribution, connected with the Markov-Pólya distribution, and which enables us to construct probabilistically the operator $L_{m,r}^{\langle\alpha\rangle}$.

Suppose that we have an urn containing identical balls of two colors : $a$ white and $b$ black. The experiment consists in drawing a ball, noting its color and returning it to the urn, together with $c$ identical balls of the same color. This process is repeated successively. Evidently the composition of the urn changes every time, except for the Bernoullian as $c = 0$.

Let $k$ be a non-negative integer such that $k \leqslant m$.

One denotes by $E_{m,r,k}$ the event that at the trial number $m-r+1$ there occur the $(m-r-k+1)$ th black ball, or the $(k-r+1)th$ white ball.

We see at once that this event occurs if :

1) in the first $m-r$ drawings we get $k$ white and $m-r-k$ black balls, while the next drawing results in a black ; we denote this event by $E'_{m,r,k}$, or if

2) in the first $m-r$ drawings we get a white ball $k-r$ times and a black ball $m-k$ times, while the next drawing leads to a white ball; we denote this event by $E''_{m,r,k}$.

It is clear that we have

$$E_{m,r,k} = \begin{cases} E'_{m,r,k} & \text{if } 0 \leqslant k < r \\ E'_{m,r,k} \cup E''_{m,r,k} & \text{if } r \leqslant k \leqslant m-r \\ E''_{m,r,k} & \text{if } m-r < k \leqslant m. \end{cases}$$

Let $T_j$ be a random variable which is either one or zero depending on the $j$-th drawing resulting in white or black. It is known [2] that the probability that the total number of white balls : $T_1 + T_2 + \ldots + T_{m-r}$ be equal with $k$ is given by

$$\binom{m-r}{k} \frac{a(a+c)\ldots(a+(k-1)c)\, b(b+c)\ldots(b+(m-r-k-1)c)}{(a+b)(a+b+c)\ldots(a+b+(m-r-1)c)}.$$

Since the $(m-r+1)th$ drawing results in a black drawing with probability $[b+(m-r-k)c]/[a+b+(m-r)c]$, we obtain the following formula

$$P(E'_{m,r,k}) = \binom{m-r}{k} \frac{a(a+c)\ldots(a+(k-1)c)b(b+c)\ldots(b+(m-r-k)c)}{(a+b)(a+b+c)\ldots(a+b+(m-r)c)}.$$

Proceeding in the same way we find that

$$P(E''_{m,r,k}) = \binom{m-r}{k-r} \frac{a(a+c)\ldots(a+(k-r)c)b(b+c)\ldots(b+(m-k-1)c)}{(a+b)(a+b+c)\ldots(a+b+(m-r)c)}.$$

Let us introduce the notations : $\frac{a}{a+b} = p$, $\frac{c}{a+b} = \alpha$. Because then we have $\frac{b}{a+b} = 1 - p = q$, we can write

$$P(E'_{m,r,k}) = \binom{m-r}{k} \frac{p(p+\alpha)\ldots(p+(k-1)\alpha)\, q(q+\alpha)\ldots(q+(m-r-k)\alpha)}{1\cdot(1+\alpha)(1+2\alpha)\ldots(1+(m-r)\alpha)} =$$

$$= \binom{m-r}{k} \frac{p^{[k,-\alpha]}\, q^{[m-r-k+1,-\alpha]}}{1^{[m-r+1,-\alpha]}}$$

and similarly

$$P(E''_{m,r,k}) = \binom{m-r}{k-r} \frac{p^{[k-r+1,-\alpha]} q^{[m-k,-\alpha]}}{1^{[m-r+1,-\alpha]}}.$$



Finally, we have

$$(3) \quad 1^{[m-r+1,-\alpha]}\, P(E_{m,r,k}) = \begin{cases} \binom{m-r}{k} p^{[k,-\alpha]} q^{[m-r-k+1,-\alpha]} & \text{if } 0 \leqslant k < r \\ \binom{m-r}{k} p^{[k,-\alpha]} q^{[m-r-k+1,-\alpha]} + \binom{m-r}{k-r} p^{[k-r+1,-\alpha]} q^{[m-k,-\alpha]} & \text{if } r \leqslant k \leqslant m-r \\ \binom{m-r}{k-r} p^{[k-r+1,-\alpha]} q^{[m-k,-\alpha]} & \text{if } m-r < k \leqslant m. \end{cases}$$

Let $X_{m,r}$ be a random variable assuming the values $k$ with the probabilities $w^{<\alpha>}_{m,r,k}(p) = P(E_{m,r,k})$, where $0 \leqslant k \leqslant m$.

First of all we calculate

$$\sum_{k=0}^{m} P(X_{m,r} = k) = \sum_{k=0}^{m} w^{<\alpha>}_{m,r,k}(p) =$$

$$= q\left[\sum_{k=0}^{m-r} \binom{m-r}{k} p^{[k,-\alpha]} (q+\alpha)^{[m-r-k,-\alpha]}\right] \Big/ (1+\alpha)^{[m-r,-\alpha]} +$$

$$+ p\left[\sum_{k=r}^{m} \binom{m-r}{k-r} (p+\alpha)^{[k-r,-\alpha]} q^{[m-k,-\alpha]}\right] \Big/ (1+\alpha)^{[m-r,-\alpha]}.$$

If in the second sum we make the change of index of summation: $k - r = j$ and if we take into account the Vandermonde convolution formula, we obtain

$$\sum_{k=0}^{m} w^{<\alpha>}_{m,r,k}(p) = [q(p+q+\alpha)^{[m-r,-\alpha]} + p(p+\alpha+q)^{[m-r,-\alpha]}]/(1+\alpha)^{[m-r,-\alpha]} = 1.$$

It is easily verified that for $r = 0$ or for $r = 1$ the probability distribution $P(X_{m,r} = k) = P(E_{m,r,k}) = w^{<\alpha>}_{m,r,k}(p)$, defined in (3), reduces to the Markov-Pólya distribution (see [2], [9], [10], [11]).

Now to each function $f : [0,1] \to \mathbb{R}$ we associate a random variable $Y^f_{m,r}$ with the probability distribution given by

$$P\left(Y^f_{m,r} = f\left(\frac{k}{m}\right)\right) = w^{<\alpha>}_{m,r,k}(p) \quad (0 \leqslant k \leqslant m).$$

The expected value of this distribution is

$$E(Y^f_{m,r}) = \sum_{k=0}^{m} w^{<\alpha>}_{m,r,k}(p) f\left(\frac{k}{m}\right).$$

We have thus constructed by a probabilistic and natural method the linear positive operator $L_{m,r}^{\langle\alpha\rangle}$ defined by

$$(L_{m,r}^{\langle\alpha\rangle} f)(x) := \sum_{k=0}^{m} w_{m,r,k}^{\langle\alpha\rangle}(x) f\left(\frac{k}{m}\right), \quad x \in [0,1].$$

We should specify that considering such an operator we shall assume — more generally — that the parameter $\alpha$ is any non-negative real number, depending in general on $m$, but such that $\alpha = \alpha(m) \to 0$ when $m \to \infty$.

For $r = 0$ or for $r=1$ this operator reduces to an operator introduced and investigated in our previous papers [8], [10], as well as in the papers [1], [3], [4], [5].

Because we can write

$$1^{[m-r+1,-\alpha]}\,(L_{m,r}^{\langle\alpha\rangle} f)\,(x) = \sum_{k=0}^{m-r}\binom{m-r}{k} x^{[k,-\alpha]}(1-x)^{[m-r-k+1,-\alpha]} f\left(\frac{k}{m}\right) + \sum_{k=r}^{m}\binom{m-r}{k-r} x^{[k-r+1,-\alpha]}(1-x)^{[m-k,-\alpha]} f\left(\frac{k}{m}\right),$$

by making in the last sum the change of index of summation $k - r = j$, we obtain

$$1^{[m-r+1,-\alpha]}\,(L_{m,r}^{\langle\alpha\rangle} f)\,(x) = \sum_{k=0}^{m-r}\binom{m-r}{k} x^{[k,-\alpha]}(1-x)^{[m-r-k+1,-\alpha]} f\left(\frac{k}{m}\right) + \sum_{j=0}^{m-r}\binom{m-r}{j} x^{[j+1,-\alpha]}(1-x)^{[m-r-j,-\alpha]} f\left(\frac{j+r}{m}\right).$$

Denoting also by $k$ the index of summation in the second sum and taking into account that

$$1^{[m-r+1,-\alpha]} = (1+\alpha)^{[m-r,-\alpha]}, \quad x^{[k+1,-\alpha]} = (x+k\alpha)^{[k,-\alpha]},$$

$$(1-x)^{[m-r-k+1,-\alpha]} = (1-x+(m-r-k)\alpha)\,(1-x)^{[m-r-k,-\alpha]},$$

we arrive finally just to the linear positive operator $L_{m,r}^{\langle\alpha\rangle}$ defined in (1) and (2).

It is easy to check that

$$(4) \qquad (L_{m,r}^{\langle\alpha\rangle} f)(0) = f(0), \; (L_{m,r}^{\langle\alpha\rangle} f)\,(1) = f(1).$$

It follows that the polynomial $L_{m,r}^{\langle\alpha\rangle} f$ is interpolatory at both sides of the interval [0,1]. This is the reason that the approximation formula

$$(5) \qquad f = L_{m,r}^{\langle\alpha\rangle} f + R_{m,r}^{\langle\alpha\rangle} f$$

has the degree of exactness $N = 1$.



Since $L_{m,r}^{\langle 0\rangle} = L_{m,r}$ is the operator investigated in detail in our paper [12] and for $x = 0$ and $x = 1$ we have the relations (4), it remains to consider the case : $\alpha > 0, 0 < x < 1$. In [13] we proved that the operator $L_{m,r}^{\langle\alpha\rangle}$ may be considered as an average of the operator $L_{m,r}$, because it can be represented in the following integral form

$$(6) \qquad (L_{m,r}^{\langle\alpha\rangle} f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}} (1-t)^{\frac{1-x}{\alpha}-1} (L_{m,r}f)(t)\, dt,$$

where $\alpha > 0$, $0 < x < 1$ and $B$ is the beta function.

Denoting by $e_j$ the monomials defined by $e_j(x) = x^j$ $(j = 0,1,2)$ and taking into consideration the relations deduced in our paper [12] :

$$(L_{m,r}e_1)\,(x) = x, \; (L_{m,r}e_2)\,(x) = x^2 + \left[1 + \frac{r(r-1)}{m}\right] \cdot \frac{x(1-x)}{m},$$

as well as the representation (6), we find that the first two moments of the probability distribution $P(X_{m,r} = k) = w_{m,r,k}^{\langle\alpha\rangle}(p)$ have the expressions

$$E(X_{m,r}) = mp, \; E(X_{m,r}^2) = \frac{1}{1+\alpha}\{m^2 p(p+\alpha) + [m + r(r-1)]pq\}.$$

Consequently for the variance of our probability distribution we find

$$\mathrm{Var}(X_{m,r}) = E(X_{m,r}^2) - E(X_{m,r})^2 = \left[1 + \alpha m + \frac{r(r-1)}{m}\right] \frac{mpq}{1+\alpha}.$$

It follows that the values of the operator $L_{m,r}^{\langle\alpha\rangle}$ for the three test functions $e_0$, $e_1$, $e_2$ are

$$(L_{m,r}^{\langle\alpha\rangle} e_0)\,(x) = 1, \; (L_{m,r}^{\langle\alpha\rangle} e_1)(x) = x,$$

$$(7) \qquad (L_{m,r}^{\langle\alpha\rangle} e_2)\,(x) = \frac{1}{1+\alpha}\left\{x(x+\alpha) + \left[1 + \frac{r(r-1)}{m}\right]\frac{x(1-x)}{m}\right\},$$

while

$$(8) \qquad (R_{m,r}^{\langle\alpha\rangle} e_2)\,(x) = -\left[1 + \alpha m + \frac{r(r-1)}{m}\right] \frac{x(1-x)}{m(1+\alpha)}.$$

According to the convergence criterion of Bohman-Korovkin-Popoviciu the results (7) permit us to state

THEOREM 1. *If* $0 \leqslant \alpha = \alpha(m) \to 0$ *as* $m \to \infty$, then for any $f \in C[0,1]$ *we have*

$$\lim_{m\to\infty} L_{m,r}^{\langle\alpha\rangle} f = f,$$

*uniformly on the interval* [0,1].

In order to evaluate the order of approximation of a function $f \in C[0,1]$ by means of $L_{m,r}^{\langle\alpha\rangle} f$, one can use the inequality (3.6) from our earlier paper [10].

In the next theorem we give the inequality which we have obtained.

THEOREM 2. *If* $f \in C[0, 1]$ *then for any* $x \in [0, 1]$ *we have*

$$|f(x) - (L_{m,r}^{<\alpha>} f)(x)| \leqslant \left(1 + \frac{1}{\gamma}\sqrt{1 + \alpha m + \frac{r(r-1)}{m}}\right)\omega\left(f; \gamma\sqrt{\frac{x(1-x)}{m(1+\alpha)}}\right),$$

*where* $\gamma$ *is an arbitrary positive constant, which can be conveniently chosen.*

Since on $[0, 1]$ we have $x(1-x) \leqslant 1/4$, if we take $\gamma = 2\sqrt{1+\alpha}$, we obtain the inequality

$$\|f - L_{m,r}^{<\alpha>} f\| \leqslant \left(1 + \frac{1}{2\sqrt{1+\alpha}}\sqrt{1 + \alpha m + \frac{r(r-1)}{m}}\right)\omega\left(f; \frac{1}{\sqrt{m}}\right),$$

which for $\alpha = 0$ and $r = 0$ (or $r = 1$) reduces to the classical inequality of T. Popoviciu [6].

With the aid of the well-known theorem of Peano one can give an integral representation for the remainder of the approximation formula (5). We can state

THEOREM 3. *If* $f \in C^2[0, 1]$ *and* $x$ *is any given point of* $[0,1]$, *then we have*

$$(9) \qquad (R_{m,r}^{<\alpha>} f)(x) = \int_0^1 G_{m,r}^{<\alpha>}(t; x) f''(t)dt,$$

*where the Peano kernel is given by*

$$G_{m,r}^{<\alpha>}(t; x) = (R_{m,r}^{<\alpha>} \varphi_x)(t),$$

*with*

$$\varphi_x(t) = (x-t)_+ = \frac{1}{2}[x - t + |x-t|],$$

*the remainder operator acting on* $\varphi_x(t)$ *as a function of* $x$.

In [13] we gave explicit expressions for the Peano kernel, which actually represents a spline function of first degree having the knots $k/m$. Because $G_{m,r}^{<\alpha>}(t; x) \leqslant 0$ for any point $(x, y) \in [0, 1]^2$, we may apply the mean value theorem to the integral occurring in (9) and we obtain

$$(10) \qquad (R_{m,r}^{<\alpha>} f)(x) = f''(\xi_{m,r}^{<\alpha>}) \int_0^1 G_{m,r}^{<\alpha>}(t; x)dt,$$

where $\xi_{m,r}^{<\alpha>} \in (0,1)$.

Taking into account that the Peano kernel is independent of the function $f$, let us insert $f = e_2$ in formula (5), with the remainder (10); we find that

$$\int_0^1 G_{m,r}^{<\alpha>}(t; x)dt = \frac{1}{2}(R_{m,r}^{<\alpha>} e_2)(x).$$



This result and formula (8) enables us to formulate

THEOREM 4. *If* $f \in C^2[0, 1]$, *then there exists a point* $\xi_{m,r}^{<\alpha>} \in (0,1)$ *such that*

$$(R_{m,r}^{<\alpha>} f)(x) = -\left[1 + \alpha m + \frac{r(r-1)}{m}\right]\frac{x(1-x)}{2m(1+\alpha)} f''(\xi_{m,r}^{<\alpha>}).$$

This result permits us to conclude that if the function $f$ is non-concave of first-order on $[0, 1]$ then we have $L_{m,r}^{<\alpha>} f \geqslant f$ on $[0,1]$, while if $f$ is convex of first-order on $[0,1]$ then we have $L_{m,r} f > f$ on the interval $(0,1)$.

Finally, we mention that the operator $L_{m,r}^{<\alpha>}$ enjoys the variation diminishing property, in the sense of I. J. Schoenberg [7], because it preserves the linear functions and if $\alpha \geqslant 0$ then the number of variations of sign of $L_{m,r}^{<\alpha>} f$ and respectively of $f$, on the interval $[0, 1]$, satisfy the inequality: $\nu(L_{m,r}^{<\alpha>} f) \leqslant \nu(f)$.

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