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AN OVERVIEW OF SEPARABLE FRACTIONAL PROGRAMMING PROBLEM

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Abstract. This paper represents a sequel to the earlier papers by the author [15 — 18], for presenting the results obtained in fractional programming. Following the classification given by Stancu-Minasian (1981d), in the paper [15] we have presented several applications of fractional programming. We have also presented the methods used for solving: a) the problems of linear fractional programming [16], b) the linear fractional programming problems with several objective functions [17] (1981b)), c) complex fractional programming [18]. In this paper we present the state of the art in separable fractional programming.

Separable programming constitues a linear programming extension for handling certain types of nonlinear functions within the framework of a general linear format. Separable programming problems are nonlinear programming problems with separable objective and constraint functions. First of all, we recall the definition of separability. The function $F(x_1,\ldots,x_n)$ is separable if $F(x_1,\ldots,x_n)=\sum_{j=1}^{n}F_j(x_j)$ i.e. it can be represented as a sum of functions each involving only one variable in its argument. Then we define the nonlinear separable programming problem.

(1) Maximize (minimize) $F(x)=\sum_{j=1}^{n}F_j(x_j)$ subject to

$$x\in D=\left\{x \mid h_i(x)=\sum_{j=1}^{n}h_{ij}(x_j)\leqslant b_i,\ i=1,\ldots,m;\ x_j\geqslant 0,\ j=1,\ldots,n\right\}$$

where the functions h_i are assumed to be separable and continuous.

The basic idea in solving the above problem is to use piecewise linear approximation to each nonlinear function of the objective function and to solve the linear programming problem so obtained. A. Charnes and C. Lemke (1954) proposed an approximation technique for minimizing nonlinear separable convex functionals subject to linear constraints. C. E. Miller (1963) generalized this approach, to include the nonlinear separable functions. Further, we recall a linearization technique due to Kortanek and Evans (1967).

THEOREM 1 (Kortanek and Evans 1967)). *Let f be a continuously differentiable function defined on the open convex set $X \subseteq R^n$. Consider the two programs : I maximize $\{f(x) : x \in C\}$ and II maximize $\{\nabla f(x^*) \cdot x : x \in C\}$ where C is a closed set contained in X, $x^* \in C$ and $\nabla f(x^*)$ is the gradient of $f(x)$ in the point x^*. Then x^* is an optimal solution for I if and only if x^* is an optimal solution for II provided either one of the following conditions holds : a) f is pseudoconcave on X : b) f is quasiconcave on X and $\nabla f(x^*) \neq 0$.*

Recently, these methods were generalized for solving separable fractional programming problems (see Almogy and Levin (1971), Anand and Swarup (1970), Arora and Aggarwal (1977), Černov and Lange (1970), Černov (1971, 1972) Gogia (1969), Jagannathan (1965a, b), Kaul and Datta (1981), Ţigan (1977, 1981) and Wadhwa (1969)).

R. Jagannathan (1965b) develops a procedure for solving the problem

2) Minimize $\left\{F(x_1, \ldots, x_n) = \sum_{j=1}^{n} (c_j/(x_j + s_j)) \; (c_j > 0,\ s_j > 0) : A, x \leqslant b x \geqslant 0\right\}$

He demonstrates the following theorem.

THEOREM 2 (Jagannathan (1965b)). *If* $F(\bar{x}) = \sum_{j=1}^{n} (c_j/(x_j + s_j))$ *and*

$$Q^0(\bar{x}) = \sum_{j=1}^{n} \lambda_j x_j^2 - \sum_{j=1}^{n} p_j x_j \text{ where } \lambda_j = -\frac{1}{s_j}\frac{\partial F_0}{\partial x_j},\ p_j = \frac{-(2x_j + s_j)}{s_j}\frac{\partial F}{\partial x_j^0} \text{ and}$$

$$\bar{x}^{0\prime} = (x_1^0, \ldots, x_n^0), \text{ then } (i)\ F(\bar{x}^0) = \sum_{j=1}^{n} (c_j/s_j) + Q^0(\bar{x}^0).$$

$$(ii)\ F(\bar{x}) \leqslant \sum_{j=1}^{n} (c_j/s_j) + Q^0(\bar{x}) \text{ for all } \bar{x} \geqslant 0, (iii)\ \frac{\partial Q^0(\bar{x})}{\partial \bar{x}^0} = \frac{\partial F(\bar{x})}{\partial \bar{x}^0}.$$

According to this theorem, R. Jagannathan reduces the solving of separable fractional programming problem (2) to a sequence of convex quadratic programming problems. Minimize $\{Q^i(x) : Ax \leqslant b ; x \geqslant 0\}$ where the solution of the k^{th} problem is used to define $Q^{(k+1)}(x)$ for the $(k+1)^{\text{th}}$ problem. The algorithm stops when the reduction in the successive values of $Q(x)$ becomes very small. We remark that due to the function $F(x_1, \ldots, x_n)$ is convex we can apply the method of A. Charnes and C. Lemke (1954) to solve the problem (1).

P. Anand and K. Swarup (1970) consider the following problem :

(3) Maximize $\left\{F(x_1, \ldots, x_n) = \sum_{j=1}^{n} [(x_j + \alpha_j)/(x_j + \beta_j)],\ (\alpha_j > 0, \beta_j > 0)\right\}$

subject to $x \in \{x \mid Ax \leqslant b,\ x \geqslant 0\}$

Here the objective function is neither concave nor convex. In order to solve this problem, they reduce it to a sequence of concave quadratic programming problems. The sequence of points solving the quadratic problems is shown to converge to a local solution of the problem (3).

The relation between the problem (3) and the concave quadratic problem is given by the following theorem.

THEOREM 3 (Anand and Swarup (1970)). *If* $F(x) = \sum_{j=1}^{n} \frac{x_j + \alpha_j}{x_j + \beta_j}$ *and*

$$Q^0(x) = -\sum_{j=1}^{n} \lambda_j x_j^2 + \sum_{j=1}^{n} [x_j(p_j - \alpha_j \lambda_j) + p_j \alpha_j] \text{ where } \lambda_j = \frac{1}{\alpha_j - \beta_j}\frac{\partial F}{\partial x_j^0} \text{ and}$$

$$p_j = -\frac{(2x_j^0 + \beta_j)}{\alpha_j - \beta_j}\frac{\partial F}{\partial x_j^0}, \text{ then } (i)\ F(x) \geqslant Q^0(x),\ (ii)\ F(x) = Q^0(x^0),$$

$$(iii)\quad \nabla F(x^0) = \nabla Q^0(x^0).$$

N.K. Gogia (1969) considers the following problem

(4) Maximize $F(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} f_j(x_j)\right) \Big/ \left(\sum_{j=1}^{n} g_j(x_j)\right)$ subject to $x \in D$ where $D = \left\{x \in R^n \mid \sum_{j=1}^{n} h_{ij}(x_j) \leqslant b_i,\ i = 1, \ldots, m,\ x_j \geqslant 0,\ j = 1, \ldots, n\right\}$ and the problem functions are assumed continuous. Each of the functions f_j, g_j and h_{ij} is approximated by a set of piecewise linear functions following C. E. Miller (1963), and the resulting linear fractional program is

(5) Maximize $F^* = \left(\sum_{j=1}^{n} \sum_{k=0}^{r_j} \lambda_{kj} f_{kj}\right) \Big/ \left(\sum_{j=1}^{n} \sum_{k=0}^{r_j} \lambda_{kj} g_{kj}\right)$ subject to $\sum_{j=1}^{n} \sum_{k=0}^{r_j} \lambda_{kj} h_{kij} \leqslant b_i,\ i = 1, \ldots, m ; \sum_{k=0}^{r_j} \lambda_{kj} = 1,\ j = 1, \ldots, n ; \lambda_{kj} \geqslant 0$ for all k, j. Solving this problem by applying simplex algorithm, with restricted basis entry in the basis such that not more than two λ_{ks} are positive we obtain the solution λ_{kj}. If the two λ_{ks} are positive then they must be adjacent. Then $x_j = \sum_{k=0}^{r_j} \lambda_{kj} x_{kj}$ will be the approximate solution of the original problem.

Ju. P. Černov and E. G. Lange (1970) give an approximative method for solving a transport problem with separable fractional function

(6) Minimize $F(x) = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij}(x_{ij})\right) \Big/ \left(\sum_{i=1}^{m} \sum_{j=1}^{n} g_{ij}(x_{ij})\right)$ where $x = (x_{ij})$ and $x \in D = \left\{x \in R \,\Big|\, \sum_{j=1}^{n} x_{ij} = a_i,\ \sum_{i=1}^{m} x_{ij} = b_j\right\}$, $R = \{x \mid \alpha_{ij} \leqslant x_{ij} \leqslant \beta_{ij}\}$.

The functions f_{ij}, g_{ij} are linearized using the δ-form (Hadley (1964)) and the new problem has the form

Minimize $\{(\sum \sum c'_{rs} y_{rs} + c'_0)/(\sum \sum c''_{rs} y_{rs} + c''_0) \mid \sum_s y_{rs} = A_r,\ \sum_r y_{rs} = B_s,\ y_{rs} \geqslant 0\}$.

and y_{rs} must satisfy a certain supplimentary condition.

Ju. P. Černov (1971) considers the following separable problem

(7) Minimize $F(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} f_j(x_j)\right) \Big/ \left(\sum_{j=1}^{n} g_j(x_j)\right)$

subject to $x \in \left\{x \,\Big|\, \sum_{j=1}^{n} h_{ij}(x_j) \leqslant b_i,\ i = 1, \ldots, n ;\ \alpha_j \leqslant x_j \leqslant \beta_j,\ j = 1, \ldots, n\right\}$.

He utilizes the same method, namely to approximate all separable functions by linear functions with a greater number of variables, and by means of it reduces the problem to linear fractional programming problem.

Vijay Wadhwa (1969) considers a class of mathematical programming problems where the objective function is the sum of separable quadratic-linear fractional functionals and the set of constraints is a convex polyhedron.

$$(8)\qquad \text{Minimize}\left\{F(x_1,\ldots,x_n)=\sum_{j=1}^{n}[(k_j+m_jx_j^2)/(x_j+c_j)]\;\middle|\; Ax=b\,;\; x\geqslant 0\right\}.$$

It is supposed that $c_j>0$, $k_j>0$, $m_j\geqslant 0$, which make the above function convex. The method proposed requires the solution of a sequence of quadratic programming problems and is based on the following theorem.

THEOREM 4 (Wadhwa (1969)). *If* $F(x)=\sum_{j=1}^{n}[(k_j+m_jx_j^2)/(x_j+c_j)]$, *where* $k_j,\ c_j>0$ *and* $m_j\geqslant 0$ *are constants and* $G_0(x)=\sum_{j=1}^{n}\alpha_jx_j^2-\sum_{j=1}^{n}\beta_jx_j$, *where* $\alpha_j=[m_j-\nabla F(x_j^0)]/c_j$, $\beta_j=[2x_j^0(k_j+m_jc_j^2)/c_j(x_j^0+c_j)^2]-\nabla F(x_j^0)$, *and* $x^0=(x_1^0,\ldots,x_n^0)\geqslant 0$, *then a)* $F(x)\leqslant G_0(x)+\sum_{j=1}^{n}(k_j/c_j)$, *for all* $x\geqslant 0$, *b)* $F(x^0)=G_0(x^0)+\sum_{j=1}^{n}(k_j/c_j)$, *c)* $\nabla F(x^0)=\nabla G_0(x^0)$.

Y. Almogy and O. Levin (1971) consider the following separable problem (9) $\text{Maximize}\left\{F(x_1,\ldots,x_n)=\sum_{i=1}^{n}[(c_i'x+\alpha_i)/(d_i'x+\beta_i)]\;\middle|\;Ax\leqslant b,\ x\geqslant 0\right\}$ where x is an n-component column vector, c_i and d_i are n-component column vectors ($i=1,\ldots,k$), A is an $m\times n$ matrix, b is an m-component column vector, and α_i, β_i are given constants.

They consider the value of each function $f_i(x_i)$ as a parameter f_i that can be varied, irrespective of whether or not it is attainable on feasible set. In this manner, the problem can be transformed into equivalent ones of maximizing multiparameter linear or concave functions subject to additional feasibility constraints. If the linear fraction's denominators are restricted to nonnegative coefficients, the problems can be transformed into those of finding a root (in the case of separability this root is unique) of a monotone-decreasing convex parametric function. When the number of terms in the objective function is equal or less than three, Y. Almogy and O. Levin give an efficient algorithm. In the special case when the coefficient matrix A consists of nonnegative elements (as in most transportation problems), Y. Almogy and O. Levin utilized the combinatorial property of the parametric presentation in outling a method of solution.

Şt. Ţigan (1977) comes to a separable fractional programming problem related to the resources dynamic distribution problem, wherein the resources utilization efficiency (mean income to the time unit) is maximized. The model is the following:

$$\text{Maximize } F(x_1,\ldots,x_n)=\left(\sum_{i=1}^{n}f_i(x_i)+\alpha\right)\Bigg/\left(\sum_{i=1}^{n}g_i(x_i)+\beta\right)$$

subject to $\sum_{i=1}^{n}w_ix_i\leqslant b$, $\alpha_i\geqslant x_i\geqslant\beta_i\geqslant 0$, x_i integers where w_i, α_i, β_i ($i=1,\ldots,n$), α, β and b are given real numbers. He gives an algorithm for solving this problem which consists in solving a finite number of ordinary dynamic programming problems.

Also, I mention a recently paper of Savita Arora and S. P. Aggarwal (1977) dealing with maximizing the sum of a finite number of separable linear concave (convex) fractional Functions. The nonconvex piecewise linear separable programming problem is the following:

$$\text{Maximize}\left\{V_s=\sum_{j=1}^{s}\frac{\gamma_j}{c_j'X_j+\alpha_j}\;\middle|\;\sum_{j=1}^{s}\bar{B}_jX_j=b,\ B_jX_j=b_j,\ j=1,\ldots,s,\ X_j\geqslant 0,\ j=1,\ldots s\right\}$$

where c_j, X_j are each $n_j\times 1$ component vectors, b and b_j are vectors with m and m_j components, $\bar{B}_j$ and B_j are $m\times n_j$ and $m_j\times n_j$ matrices, γ_j and α_j are scalar constants. A decomposition principle is derived with the dynamic programming approach to solving this problem. This results in a series of parametric quadratic fractional subprogrammes whose recursive solution yields the solution to the original problem.

Finally, I give a recent paper of R. N. Kaul and Neelam Datta (1981) which considers the following nonlinear fractional programming problem.

$$\text{Maximize } F(x)=\left(\sum_{j=1}^{n_1}f_j(x_j)+\sum_{j=1}^{n_2}h_j(y_j)\right)\Bigg/\sum_{j=1}^{n_1}g_j(x_j)\ \text{subject to}\ \sum_{j=1}^{n_1}e_{ij}(x_j)+\sum_{j=1}^{n_2}a_{ij}y_j\leqslant b_i,\ i=1,\ldots,m\,;\ x_j\geqslant 0,\ j=1,\ldots,n_1\,;\ y_j\geqslant 0,\ j=1,\ldots,n_2$$

where the functions $f_j(x_j)$ ($j=1,\ldots,n_1$) are concave and $h_j(y_j)$ ($j=1,\ldots,n_2$) $g_j(x_j)$ ($j=1,\ldots,n_1$) and e_{ij} ($i=1,\ldots,m$; $j=1,\ldots,n_1$) are convex and $g_j(x_j)$ are assumed to be positive over the feasibility region.

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