MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 14, N° 1, 1985, pp. 97—103

THE DIRICHLET PROBLEM FOR A CLASS OF STRONGLY NONLINEAR ELLIPTIC EQUATIONS

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the boundary $\partial\Omega$,

(1) $$Au = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, Du, \ldots, D^m u)$$

a differential expression of divergence form, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $Du = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$, $D^m u = \left\{D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1}, \ldots, \partial x_n^{\alpha_n}} \,\Big|\, |\alpha| = m\right\}$, and $W^{m,p}(\Omega)$ the Sobolev space of functions $\{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), |\alpha| \leq m\}$ with the norm

(2) $$\|u_{m,p}\| = \left\{\sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u(x)|^p \, dx\right\}^{1/p}.$$

Let finally $W_0^{m,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$. It is known, that $u \in W_0^{m,p}(\Omega)$ if and only if $u \in W^{m,p}(\Omega)$ and $D^\alpha u|_{\partial\Omega} = 0 \; \forall \; |\alpha| \leq m-1$.

The boundary value problem studied in this paper is

(3) $$Au + g(x, u) = f \text{ in } \Omega, \; D^\alpha u|_{\partial\Omega} = 0 \text{ for } |\alpha| \leq m-1,$$

where $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ are given functions. It is supposed that $f \in L^q(\Omega)$ $(p^{-1} + q^{-1} = 1)$. We say that $u \in W_0^{m,p}(\Omega)$ is a solution of the boundary value problem (3), if for this u the integrals

(4) $$\int_\Omega A_\alpha(x, u, Du, \ldots, D^m u) \, D^\alpha v \, dx, \; |\alpha| \leq m, \int_\Omega g(x, u) v \, dx$$

exist for all $v \in W_0^{m,p}(\Omega)$ and for every $v \in C_0^\infty(\Omega)$ we have

(5) $$\sum_{|\alpha| \leq m} \int_\Omega A_\alpha(x, u, Du, \ldots, D^m u) \, D^\alpha v \, dx + \int_\Omega g(x, u) v \, dx = \int_\Omega f v \, dx.$$

To assure the solvability of the problem (3), we need the following assumptions upon the functions A_α and g:

I. The functions $A_\alpha : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfy the Caratheodory conditions, i.e. $A(x, \xi)$ is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ ($N = \text{card}\{\alpha \in \mathbb{Z}_+^n \mid |\alpha| \leq m\}$, continuous in ξ for a.e. $x \in \Omega$. There exist two constants

$p \in (1, \infty)$, $C_1 > 0$ and a function $k_1 \in L^q(\Omega)$ such that

$$|A_\alpha(x, \xi)| \leqslant C_1 |\xi|^{p-1} + k_1(x) \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^N. \tag{6}$$

II. If $x \in \Omega$, $\xi' = (\eta, \zeta')$, $\xi'' = (\eta, \zeta'')$, $\zeta' \neq \zeta''$ we have

$$\sum_{|\alpha|=m} [A_\alpha(x, \xi') - A_\alpha(x, \xi'')](\zeta'_\alpha - \zeta''_\alpha) > 0, \tag{7}$$

where $\eta \in \mathbb{R}^{N_1}$ ($N_1 = \operatorname{card}\{\alpha \in \mathbb{Z}_+^n \mid |\alpha| \leqslant m - 1\}$ stands for the derivatives of order less then m, and ζ', ζ'' stands for the derivatives of order m.

III. There exists a constant $C_2 > 0$ and a function $k_2 \in L^1(\Omega)$ such that

$$\sum_{|\alpha| \leqslant m} A_\alpha(x, \xi)\xi_\alpha \geqslant C_2 |\xi|^p - k_2(x) \quad \forall x \in \Omega,\ \xi \in \mathbb{R}^N. \tag{8}$$

IV. The function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a positive constant m such that

$$g(x, r)\, r + m \geqslant 0 \quad \forall x \in \overline{\Omega},\ r \in \mathbb{R}. \tag{9}$$

V. There exist two constants $C_3 > 0$, $a > 0$ and a continuous non-decreasing function $h : \mathbb{R} \to \mathbb{R}$ such that for $|r| \geqslant a$ we have

$$C_3(|h(r)| + |r|^{p-1} + 1) \leqslant |g(x, r)| \leqslant |h(r)| \quad \forall x \in \overline{\Omega}. \tag{10}$$

The main result of the paper is :

THEOREM. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $A\,u$ the differential expression given by* (1), *for which the assumptions* **I, II, III** *are fulfilled and $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ a function which satisfies the assumptions* **IV** *and* **V**. *If $f \in L^q(\Omega)$, then there exists at least one solution $u \in W_0^{m,p}(\Omega)$ of the* **Dirichlet problem** (3).

In the proof of the theorem we denote $V = W_0^{m,p}(\Omega)$ and use the symbol $\rightharpoonup$ for the weak convergence and $\to$ for the strong convergence. At first we establish some auxiliary results.

PROPOSITION 1. *Let $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the condition* **IV** *and*

$$g_k(x, r) = \begin{cases} -k & \text{if } g(x, r) < -k \\ g(x, r) & \text{if } -k \leqslant g(x, r) \leqslant k \\ k & \text{if } g(x, r) > k \end{cases} \qquad k = 1, 2, \ldots \tag{11}$$

If $\{u_k\} \subset V$ is a sequence for which $u_k \rightharpoonup u$ in V, $\int_\Omega g_k(x, u_k)u_k \mathrm{d}x \leqslant K = \text{const.}$, then

a) $g(\cdot, u) \in L^1(\Omega)$, $u\, g(\cdot, u) \in L^1(\Omega)$,

b) there exists a subsequence $\{u_{k_j}\} \subset \{u_k\}$ for which

$$\int_\Omega g_{k_j}(x, u_{k_j})\mathrm{d}x \to \int_\Omega g(x, u)\, \mathrm{d}x, \int_\Omega g_{k_j}(x, u_{k_j}) v \mathrm{d}x \to \int_\Omega g(x, u)\, v \mathrm{d}x \quad \forall v \in C(\overline{\Omega}),$$

c) for the subsequence $\{u_{k_j}\}$ we have

$$\int_\Omega g(x, u)\, u\, \mathrm{d}x \leqslant \lim\inf \int_\Omega g_{k_j}(x, u_{k_j})\, u_{k_j} \mathrm{d}x.$$

The proof of this proposition is the same as the proof of Lemma 4 in the paper [4].

PROPOSITION 2. *Let $H : \mathbb{R} \to \mathbb{R}$ be a continuous convex function, $H(0) = 0$ and $u \in V$ for which $H(u) \in L^1(\Omega)$. Then there exists a sequence $\{v_j\} \subset C_0^\infty(\Omega)$ with the following properties :*

a) $v_j \rightharpoonup u$ in V, $v_j(x) \to u(x)$ almost everywhere in Ω,

b) $H(v_j) \to H(u)$ in $L^1(\Omega)$, $H(v_j(x)) \to H(u(x))$a.e. in Ω,

c) there exists a function $k_3 \in L^1(\Omega)$ such that $H(v_j(x)) \leqslant k_3(x)$.

The proof of Proposition 2 is to be found in [1].

PROPOSITION 3. *If the function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the assumptions* **IV** *and* **V** *then there exists a constant $C_4 > 0$ such that*

$$|g(x, r)\, s| \leqslant C_4[H(s) + r\, h(r) + 1] \quad \forall r, s \in \mathbb{R}, \tag{12}$$

where h is the function from condition **V** *and* $H(r) = \int_0^r h(\tau)\, \mathrm{d}\tau$.

Proof. It is easy to verify that the function h in the assumption V may be replaced by a continuous nondecreasing odd function. In this case H is a continuous convex even function, $H(0) = 0$, $H(r) \geqslant 0$, $H(r) \leqslant rh(r)$ $\forall r \in \mathbb{R}$. If $r, s \in [-a, a]$, then there exists a constant $\gamma_4 > 0$ such that

$$|g(x, r)s| \leqslant \gamma_4[H(s) + r\, h(r) + 1]$$

since g is a bounded function upon $\overline{\Omega} \times [-a, a]$.

For $|r|$, $|s| \in (a, \infty)$ we use the monotonicity of h and the convexity of H. We have $H'(r) = h(r)$, $h(r)(s - r) \leqslant H(s) - H(r) \leqslant H(s)$, $\forall r, s \in \mathbb{R}$, that is

$$h(r)s \leqslant H(s) + r\, h(r). \tag{13}$$

For $r, s \in (a, \infty)$ or $r, s \in (-\infty, -a)$ it results from the assumption **V** that

$$g(x, r)s = |g(x, r)|\, |s| \leqslant |h(r)|\, |s| = h(r)s \leqslant H(s) + r\, h(r),$$

and for $r \in (a, \infty)$, $s \in (-\infty, -a)$

$$|g(x, r)\, s| \leqslant h(r)|s| \leqslant H(|s|) + rh(r) = H(s) + r\, h(r).$$

Finally for $r \in (-\infty, -a)$, $s \in (a, \infty)$ we have

$$|g(x, r)s| \leqslant |h(r)|\, s = -h(r)s = h(-r)s \leqslant H(s) + (-r)h(-r) = H(s) + rh(r).$$

Denoting $C_4 = \max(\gamma_4, 1)$ we obtain the statement of Proposition 3.

Proof of the theorem : Observ that under assumption **I** $A_\alpha(., u, \ldots, D^m u) \in L^q(\Omega)$ for any $u \in V = W_0^{m,p}(\Omega)$, and thus $a_\alpha(u, v) = \int_\Omega A_\alpha(x, u, \ldots, D^m u)\, D^\alpha v \, dx$ is well defined, $a_\alpha : V \times V \to \mathbb{R}$ is a linear, bounded functional of v on V. Thus there exists a mapping $A : V \to V^*$ (V^* is the dual space of V) such that

$$(14) \qquad \int_\Omega \sum_{|\alpha| \leqslant m} A_\alpha(x, u, \ldots, D^m u)\, D^\alpha v \, dx = (Au, v) \quad \forall u, v \in V.$$

Assumptions **I**, **II**, and **III** assure that $A : V \to V^*$ is a continuous coercive pseudomonotone mapping and maps bounded sets of V into bounded sets of V^* [2].
(We recall that an operator $A : V \to V^*$ is said to be pseudomonotone if A is continuous from finite dimensional subspaces of V to the weak topology of V^* and for any sequence $\{u_k\} \subset V$, $u_k \rightharpoonup u$ for which $\limsup\,(Au_k, u_k - u) \leqslant 0$ it follows that $Au_k \rightharpoonup Au$ in V^* and $(Au_k, u_k) \to (Au, u)$).

We shall study now the influence of the perturbing term $\int_\Omega g(x, u) v \, dx$ in the equation (5). Using inequalities (10), we see that g is unbounded, when $|r| \to \infty$, so the integral $\int_\Omega g(x, u)\, v dx$ may be either convergent or divergent for $u, v \in V$. In the equation (3) we substitute g by the function $g_k (k = 1, 2, \ldots)$ defined by (11). Using assumptions **I** – **IV** we can prove (as in the case of mapping A) that the operator $u \mapsto Au + g_k(., u)$ from V to V^* is a continuous coercive bounded pseudomonotone mapping.

We study now the equation

$$(15) \qquad Au + g_k(x, u) = f \qquad u \in V$$

with a given $f \in V^*$. The mapping $Au + g_k(x, u)$ being continuous coercive bounded and pseudomonotone, it follows [3] that for each $k \in \mathbb{N}$ there exists $u_k \in V$ such that

$$(16) \qquad (Au_k, v) + \int_\Omega g_k(x, u_k)\, v \, dx = (f, v) \ \forall v \in V.$$

Particulary for $v = u_k$ we have

$$(17) \qquad (Au_k, u_k) + \int_\Omega g_k(x, u_k)\, u_k dx = (f, u_k) \quad k = 1, 2, \ldots.$$

The sets $\{\|u_k\|_V\}_{k=1}^\infty$ and $\{\|Au_k\|_{V^*}\}_{k=1}^\infty$ are bounded. Indeed, from (17) and $g_k(x, u_k)u_k + m > 0$ it follows that

$$\frac{|(Au_k, u_k)|}{\|u_k\|} = \left| \frac{(f, u_k)}{\|u_k\|} - \frac{1}{\|u_k\|} \int_\Omega g_k(x, u_k)\, u_k dx \right| \leqslant \|f\| + \frac{m \text{ mes } \Omega}{\|u_k\|}.$$

If $\{\|u_k\|\}_{k=1}^\infty$ would be unbounded, then we would have a subsequence $\{u_{k_j}\} \subset \{u_k\}$ such that $\dfrac{(Au_{k_j}, u_{k_j})}{\|u_{k_j}\|} \to \infty$ and $\|f\| + \dfrac{m \text{ mes } \Omega}{\|u_{k_j}\|} \to 0$ because A is a coercive mapping. Since A maps bounded sets into bounded sets, it follows that $\{\|Au_k\|_{k=1}^\infty\}$ is bounded.

Using the reflexivity of the spaces V and V^* we can select a subsequence $\{u_{k_i}\} \subset \{u_k\}$ such that $\{u_{k_i}\}$ and $\{Au_{k_i}\}$ would be weakly convergent sequences in V resp V^*. Let

$$u_{k_i} \rightharpoonup u, \ Au_{k_i} \rightharpoonup w.$$

For the shortness of the writing we also denote the subsequence $\{u_{k_i}\}$ by $\{u_k\}$. We have then for any k : $\|u_k\|_V \leqslant \gamma_1$, $\|Au_k\|_{V^*} \leqslant \gamma_2$ and m mes $\Omega \leqslant$
$\leqslant \int_\Omega g_k(x, u_k) u_k \, dx = (f, u_k) - (Au_k, u_k) \leqslant \|f\| \cdot \|u_k\| + \|Au_k\| \cdot \|u_k\| \leqslant \gamma_3,$

where γ_1, γ_2 and γ_3 are suitable chosen constants.

In order to apply the pseudomonotonicity of A, we try to show that $\limsup\,(Au_k, u_k - u) \leqslant 0$. If $v \in V \cap L^\infty(\Omega)$, we have

$$(Au_k, u_k - u) = (Au_k, u_k - v) + (Au_k, v - u) =$$

$$= (f, u_k - v) - \int_\Omega g_k(x, u_k)\, u_k dx + (Au_k, v - u).$$

But

$$(18) \qquad u_k \rightharpoonup u, \ Au_k \rightharpoonup w, \quad \int_\Omega g_k(x, u_k)\, u_k dx \leqslant \gamma_3,$$

consequently, according to Proposition 1 $ug(., u) \in L^1(\Omega)$ and $\int_\Omega g(x, u) u dx \leqslant$
$\leqslant \liminf \int_\Omega g_k(x, u_k) u_k \, dx$ at least for a subsequence of $\{u_k\}$. Thus it follows

$$(19) \quad \limsup\,(Au_k, u_k - u) \leqslant (f - w, u - v) - \int_\Omega g(x, u)\,(u - v) dx.$$

For the function $H(r) = \int_0^r h(\tau)\, d\tau$ we have

$$(20) \qquad H(u) \leqslant h(u)\, u = |h(u)| \cdot |u| \leqslant C_3 |u|\, [\,|g(x, u)| + |u|^{p-1} + 1],$$

therefore $H(u)$ is dominated by an integrable function on Ω, so $H(u) \in L^1(\Omega)$.

Let $\{v_j\} \subset C_0^\infty(\Omega)$ be the sequence described in Proposition 2. By the inequality (12) we have

$$g(x, u)v_j + m \leqslant [g(x, u)v_j + m]^+ \leqslant C_5[H(v_j) + uh(u) + 1],$$

where $[g(x, u)v_j + m]^+$ denotes the non-negative part of $g(x, u)v_j + m$. The right side of the last inequality is an integrable function, $[g(x, u)v_j + m]^+$ converges almost everywhere to $g(x, u)u + m$, consequently by the Lebesgue dominated convergence theorem

$$\int_\Omega [g(x, u)v_j + m]^+ \, dx \to \int_\Omega g(x, u) \, u dx + m \text{ mes } \Omega. \tag{21}$$

Choosing $v = v_j$ in (19) we obtain

$$\lim \sup \, (Au_k, \, u_k - u) \leqslant (f - w, \, u - v_j) - \int_\Omega g(x, u) \, (v_j - u) dx.$$

But $v_j \to u$ in V, consequently $(f - w, u - v_j) \to 0$ and on the other hand

$$\int_\Omega g(x, u) \, (v_j - u) dx \leqslant \int_\Omega [g(x, u) \, v_j + m] \, dx - \int_\Omega [g(x, u)u + m] \, dx \leqslant$$

$$\leqslant \int_\Omega [g(x, u) \, v_j + m]^+ \, dx - \int_\Omega g(x, u) \, u dx - m \text{ mes } \Omega \to 0.$$

We have after all

$$\lim \sup \, (Au_k, u_k - u) \leqslant 0, \; u_k \rightharpoonup u, \; Au_k \rightharpoonup w.$$

Since A is pseudomonotone, it follows that $w = Au$ and $(Au_k, u_k) \to (Au, u)$. The equalities (16), Proposition 1 and $Au_k \rightharpoonup Au$ give then

$$(Au_k, v) = (f, v) - \int_\Omega g_k(x, u_k) v dx \to (f, v) - \int_\Omega g(x, u) \, v dx \quad \forall v \in V.$$

If $f \in L^q(\Omega)$, it follows that

$$(Au, v) + \int_\Omega g(x, u) \, v dx = \int_\Omega f(x) \, v(x) \, dx \quad \forall v \in V, \tag{22}$$

thus u is a solution of problem (3).

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Received 3.II, 1984

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