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THE JENSEN INEQUALITY: REFINEMENTS AND APPLICATIONS

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1. Let I be a finite nonempty set of positive integers.

If $p_i > 0$, $x_i \in [a, b]$, $i \in I$ and if f is a real function defined on $[a, b]$, let us denote:

(1) $$F(I,f) = (\sum_{i\in I} p_i) f\left(\frac{\sum_{i\in I} p_i x_i}{\sum_{i\in I} p_i}\right) - \sum_{i\in I} p_i f(x_i)$$

If f is convex on $[a, b]$, then according to the Jensen inequality:

(2) $$F(I,f) \leqslant 0$$

P. M. Vasić and Z. Mijalković have proved in [4] that if I, J are finite nonempty sets of positive integers, $I \cap J = \emptyset$, $p_k > 0$, $x_k \in [a, b]$, $k \in I \cup J$, and if f is convex on $[a, b]$, then:

(3) $$F(I \cup J, f) - F(I,f) - F(J,f) \leqslant 0$$

In this paper we shall find bounds for $F(I, f)$ and for $F(I \cup J, f) - F(I,f) - F(J,f)$, if f is in $C^2[a, b]$. As applications, we shall improve some inequalities obtained by W. N. Everitt [1] and P. M. Vasić, Z. Mijalković [4]; also, we shall extend some results of I. Raşa [3].

2. Let us denote now

$$P_I = \sum_{i\in I} p_i, \quad \sigma_I(x, p) = \frac{1}{P_I^2} \sum_{\substack{i,j\in I\\ i<j}} p_i p_j (x_i - x_j)^2$$

$$A_I(x, p) = \frac{1}{P_I} \sum_{i\in I} p_i x_i; \quad G_I(x, p) = (\prod_{i\in I} x_i^{p_i})^{1/P_I}, \ (x_i > 0)$$

$$\delta_{I,J}(x, p) = \frac{P_I P_J}{P_{I\cup J}} (A_I(x, p) - A_J(x, p))^2$$

$$m_2(f) = \min_{t\in[a,b]} f''(t), \quad M_2(f) = \max_{t\in[a,b]} f''(t) \ (f \in C^2[a, b])$$

THEOREM 1. *If I is a finite nonempty set of positive integers, $p_i > 0$, $x_i \in [a, b]$, $i \in I$, and if f is a function in $C^2[a, b]$, then*

$$(4) \qquad -\frac{1}{2} M_2(f).\, P_I\, \sigma_I(x, p) \leqslant F(I, f) \leqslant -\frac{1}{2} m_2(f)\, P_I\, \sigma_I(x, p)$$

Proof. We consider the functions $f_1 f_2 : [a, b] \to \mathbf{R}$

$$f_1(t) = f(t) - \frac{1}{2} m_2(f)\, t^2,\quad f_2(t) = \frac{1}{2} M_2(f)\, t^2 - f(t)$$

which are convex on $[a, b]$. According to (2) we have $F(I, f_k) \leqslant 0$, $k = 1, 2$. It follows

$$(5) \qquad \frac{1}{2} M_2(f)\, F(I, t^2) \leqslant F(I, f) \leqslant \frac{1}{2} m_2(f)\, F(I, t^2)$$

But $F(I, t^2) = -P_I \sigma_I(x, p)$; thus (5) implies (4).

THEOREM 2. *If I, J are finite nonempty sets of positive integers, $I \cap J = \emptyset$, $p_k > 0$, $x_k \in [a, b]$, $k \in I \cup J$, and if f is a function in $C^2[a, b]$, then*

$$(6) \qquad -\frac{1}{2} M_2(f)\, \delta_{I,J}(x, p) \leqslant F(I \cup J, f) - F(I, f) - F(J, f) \leqslant$$

$$\leqslant -\frac{1}{2} m_2(f)\, \delta_{I,J}(x, p)$$

Proof. Let f_1, f_2 be as above. According to (3) we have

$$(7) \qquad \frac{1}{2} M_2(f)\, (F(I \cup J, t^2) - F(I, t^2) - F(J, t^2)) \leqslant F(I \cup J, f) - F(I, f) -$$

$$- F(J, f) \leqslant \frac{1}{2} m_2(f)\, (F(I \cup J, t^2) - F(I, t^2) - F(J, t^2)).$$

It is easy to verify that $F(I \cup J, t^2) - F(I, t^2) - F(J, t^2) = \delta_{I,J}(x, p)$. Thus, from (7) we deduce (6)

3. If $0 < a < b$, $f : [a, b] \to \mathbf{R}$, $f(t) = \ln t$, then $m_2(f) = -\dfrac{1}{a^2}$, $M_2(f) = -\dfrac{1}{b^2}$. From (4) we obtain

$$(8) \qquad \exp \frac{\sigma_I(x, p)}{2b^2} \leqslant \frac{A_I(x, p)}{G_I(x, p)} \leqslant \exp \frac{\sigma_I(x, p)}{2a^2}.$$

The right-hand inequality can be improved (see I. Raşa [2], [3]):

$$(9) \qquad \exp \frac{\sigma_I(x, p)}{2b^2} \leqslant \frac{A_I(x, p)}{G_I(x, p)} \leqslant 1 + \frac{\sigma_I(x, p)}{2a^2}$$

If we apply Theorem 2 to the above function f, we obtain

$$(10) \qquad \left(\frac{A_I(x, p)}{G_I(x, p)}\right)^{P_I} \left(\frac{A_J(x, p)}{G_J(x, p)}\right)^{P_J} \exp \frac{1}{2b^2}\, \delta_{I,J}(x, p) \leqslant \left(\frac{A_{I \cup J}(x, p)}{G_{I \cup J}(x, p)}\right)^{P_{I \cup J}} \leqslant$$

$$\leqslant \left(\frac{A_I(x, p)}{G_I(x, p)}\right)^{P_I} \left(\frac{A_J(x, p)}{G_J(x, p)}\right)^{P_J} \exp \frac{1}{2a^2}\, \delta_{I,J}(x, p)$$

This in an improvement of the following Popoviciu type inequality obtained by P. M. Vasić and Z. Mijalković in [4]:

$$(11) \qquad \left(\frac{A_I(x, p)}{G_I(x, p)}\right)^{P_I} \left(\frac{A_J(x, p)}{G_J(x, p)}\right)^{P_J} \leqslant \left(\frac{A_{I \cup J}(x, p)}{G_{I \cup J}(x, p)}\right)^{P_{I \cup J}}$$

4. If $0 < a < b$, $f : [a, b] \to \mathbf{R}\, f(t) = e^t$, then $m_2(f) = e^a$, $M_2(f) = e^b$. Using (4) and substituting a by $\ln a$, x_i by $\ln x_i$, b by $\ln b$, we get

$$(12) \qquad \frac{a}{2} \sigma_I(\ln x, p) \leqslant A_I(x, p) - G_I(x, p) \leqslant \frac{b}{2} \sigma_I(\ln x, p).$$

Using (6) and the same substitutions, we obtain

$$(13) \qquad P_I(A_I(x, p) - G_I(x, p)) + P_J(A_J(x, p) - G_J(x, p)) +$$

$$+ \frac{a}{2} \delta_{I,J}(\ln x, p) \leqslant P_{I \cup J}(A_{I \cup J}(x, p) - G_{I \cup J}(x, p)) \leqslant P_I(A_I(x, p) -$$

$$- G_I(x, p)) + P_J(A_J(x, p) - G_J(x, p)) + \frac{b}{2} \delta_{I,J}(\ln x, p).$$

This is an improvement of the following Rado type inequality obtained by W. N. Everitt in [1]:

$$(14) \qquad P_I(A_I(x, p) - G_I(x, p)) + P_J(A_J(x, p) - G_J(x, p)) \leqslant$$

$$\leqslant P_{I \cup J}(A_{I \cup J}(x, p) - G_{I \cup J}(x, p))$$

Remark. The inequalities (8), (10), (12), (13) are generalizations of some corresponding inequalities from [3] obtained by the same methods in the case $I = \{1, \ldots, n-1\}$, $J = \{n\}$, $p_1 = \ldots = p_n = 1$.

5. If $0 < a < b$, $c \in \mathbf{R}$, $f : [a, b] \to \mathbf{R}$, $f(t) = t^c$, then for $c \in [0, 1] \cup [2, +\infty)$ we have $m_2(f) = c(c-1)\, a^{c-2}$, $M_2(f) = c(c-1)\, b^{c-2}$, and for $c \in (-\infty, 0) \cup (1, 2)$ we have $m_2(f) = c(c-1)\, b^{c-2}$ $M_2(f) = c(c-1)\, a^{c-2}$. Theorem 1 implies.

$$(15) \qquad \frac{c(c-1)}{2} a^{c-2}\, \sigma_I(x, p) \leqslant A_I(x^c, p) - (A_I(x, p))^c \leqslant$$

$$\leqslant \frac{c(c-1)}{2} b^{c-2}\, \sigma_I(x, p)$$

if $c \in [0,1] \cup [2, +\infty)$, and

$$(16) \qquad \frac{c(c-1)}{2} b^{c-2} \sigma_I(x, p) \leqslant A_I(x^c, p) - (A_I(x, p))^c \leqslant \leqslant \frac{c(c-1)}{2} a^{c-2} \sigma_I(x, p) \text{ if } c \in (-\infty, 0) \cup (1, 2).$$

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