MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 14, N° 2, 1985, pp. 159 – 170

A-POSTERIORI BOUNDS FOR STEFFENSEN-LIKE METHODS

RUDOLF L. VÖLLER
(Düsseldorf)

1. Introduction. Let X be a Banach space, D an open subset of X and $F : D \to X$ a nonlinear operator. Given the problem

(1) $$F(x) = 0$$

we are interested in solutions $x^* \in D$ for the equations (1) and if the exists at least one solution, we want to compute it approximately. In this paper we shall do this by Steffensen-like methods, which are modifications of the iteration method, which has been established for one-dimensional fixed-point-equations by J. F. Steffensen in 1933 [16]. This method has been generalized by Chen [5] and Ul'm [17] for equations in Banach spaces in the following way. Given $x_0 \in D$ then for all $n \in N_0$ compute

(2) $$\tilde{x}_n = x_n - F(x_n)$$

solve

(3) $$J(\tilde{x}_n, x_n)c_n = -F(x_n)$$

and take

(4) $$x_{n+1} = x_n - c_n$$

where $J(.,.)$ is a generalized divided – difference – operator. The advantage of this methods is that it converges quadratically as well as Newton's method but the computation of the derivative is avoided. Many authors have given convergence-proofs for Steffensen's method – local convergence has been proved by J. W. Schmidt [13], semilocal convergence by Chen [5], Ul'm [17], Bel'tyukov [4], Kippel' [10], Johnson and Scholz [8], Balász [1] and for the one-dimensional case by Baptist [3] – , but no a-posteriori bound has been established, which approximates the error in a numerically sufficient way. Only in the case that Steffensen's method converges monotonically satisfying a-posteriori bounds are known – comp. Mönch [11], Hofmann [7] and Schneider [15] – but then the operator F has to fulfill some additional conditions.

A further generalisation for equations in Fréchetspeces has been done by Balász and Goldner [2].

Since the way of calculating $\tilde{x}_n$ by (2) might be crucial, in this paper we take

$$\tilde{x}_n = x_n - \lambda_n F(x_n) \tag{5}$$

with a certain parameter $\lambda_n \in \mathbb{R}^+$, so that $\tilde{x}_n$ is not "too far away" from x_n and present a new semilocal-convergence-theorem for the Steffensen-like method (5), (3), (4). The proof is very similar to that given by B. Döring [6] to establish an a-posteriori bound for Newton's method. Furthermore the conditions are somewhat weaker than those in [1], [4], [5], [8], [10], since we use "consistent approximations" (see [12] and [14]) instead of "generalized divided differences", so that our theorem has a wider range of application. The goodness of the a-posteriori bound is shown by two examples.

2. Preliminaries. Let X, D be given as in section 1, Y be a Banach space and F Gateaux-differentiable operator from D into Y. If there exists an operator $J : D \times D \to B(X, Y)$, where $B(X, Y)$ is the set of all linear, continuous maps from X to Y, such that

$$\forall x, y, z \in D: \ \|J(x, y) - F'(z)\| \leqslant C_1 \|x - z\| + C_2 \|y - z\| \tag{7}$$

with $C_1, C_2 \in \mathbb{R}^+$ independent of x, y, z, then J is called a consistent approximation of F'.

It is easily seen that if there exists a consistent approximation J of F' on D, then

$$\forall x \in D: F'(x) = J(x, x) \tag{8}$$

and

$$\forall x, y \in D: \ \|F'(x) - F'(y)\| \leqslant [C_1 + C_2] \|x - y\| \tag{9}$$

that means F' is Lipschitz-continuous. Furthermore

$$\forall x, y, z \in D: \ \|J(x, y) - J(y, z)\| \leqslant C_1 \|x - y\| + C_2 \|y - z\| \tag{10}$$

On the other hand, if an operator $J : D \times D \to B(X, Y)$ exists satisfying (8) and (10), then J is a consistent approximation of F' and F' is Lipschitz-continuous.

From (9) it follows also (see [12], Th. 3.2.12) that if D is convex, then

$$\forall x, y \in D: \| F(x) - F(y) - F'(y)(x - y)\| \leqslant \frac{C_1 + C_2}{2} \| x - y\|^2. \tag{11}$$

Using the concept of consistent approximation J we now solve (1) by the Steffensen-like method (5), (3), (4) and call the sequence $\{x_n\}n \in \mathbb{N}_0$ the Steffensen-iteration-sequence (SIS).

3. Main result. Let X, D, F be given as is section 1. Furthermore let D be convex, F be continuous and J be a consistent approximation of F' on D. Then we can prove the following

THEOREM: *If there exist elements $\tilde{x}_0, x_0 \in D$, such that*

(V1) $\quad J(\tilde{x}_0, x_0)^{-1}$ *exists and* $\|\tilde{x}_0 - x_0\| \leqslant \zeta$

with $\zeta \geqslant \| J(\tilde{x}_0, x_0)^{-1} F(x_0)\|$

(V2) $\quad \| J_0(\tilde{x}_0, x_0)^{-1} \| \leqslant \beta$

(V3) $\quad \eta : = \beta K \zeta \leqslant \frac{1}{4}$

with $K \geqslant \max\{C_1 + C_2, \ 2C_1\}$, C_1, C_2 *from* (7)

where $x, y \in S := \{x \in D \mid \ \|x - x_0\| \leqslant \zeta\}$

then

(B1) (*SIS*) *determinated by* (7) *with* λ_n *from* (12) *exists and lies in* S *if* $F(x_n) \neq 0$—*otherwise the iteration stops and* $x_n = x^*$ *is a solution of* (1).

(B2) (*SIS*) *converges quadratically against a solution* $x^* \in S$ *of* (1).

(B3) *We have the following error bounds*

$$\forall n \in \mathbb{N}_0 : \| x_n - x^* \| \leqslant \left(\frac{1}{2}\right)^{n-1} \zeta \ (\textit{a-priori bound } 1)$$

$$\forall n \in \mathbb{N}_0 : \| x_n - x^* \| \leqslant \frac{(4\eta)^{2^n - 1}}{2^{n-1}} \ (\textit{a-priori bound } 2)$$

$$\forall n \in \mathbb{N} : \| x_n - x^* \| \leqslant \frac{2\beta_{n-1} K \zeta_{n-1}^2}{1 - 2\eta_{n-1} + \sqrt{1 - 4\eta_{n-1}}}$$

(a-posteriori bound)

with

$$\eta_0 : = \eta, \ \beta_0 : = \beta$$

$$\zeta_n : = \| x_{n+1} - x_n \| \ n \in \mathbb{N}_0 \tag{12}$$

$$\beta_n : = \frac{\beta_{n-1}}{1 - 2\eta_{n-1}}, \quad \eta_n = \beta_n K \zeta_n, \quad n \in \mathbb{N}$$

$$\lambda_n = \min \left\{ \frac{2}{3 \|J(\tilde{x}_{n-1}, x_{n-1})\|}, \ \frac{\| x_n - x_{n-1}\|}{\|F(x_n)\|} \right\}$$

(B4) x^* *is the only solution of* (1) *in* S *and*

$$\| x^* - x_0 \| \leqslant \frac{1 - \sqrt{1 - 4\eta}}{2\eta} \zeta \tag{13}$$

Proof: (B1) We show by induction

$(A1)_n : \ J_n^{-1} : = J(\tilde{x}_n, x_n)^{-1}$ exists and $\|J_n^{-1}\| \leqslant \beta_n$

$(A2)_n : \ \eta_n \leqslant \frac{1}{4}$

$(A3)_n : S_n := \{x \in D \mid \|x - x_n\| \leqslant 2\zeta_n\} \subset \ldots \subset S_0 := S$

$(A4)_n : \|\tilde{x}_n - x_n\| \leqslant \zeta_n$

$(A1)_0$, $(A2)_0$, $(A3)_0$, $(A4)_0$ are true because of $(V1)$, $(V2)$ and $(V3)$. Therefore we suppose that $(A1)_{k-1}$, $(A2)_{k-1}$, $(A3)_{k-1}$ and $(A4)_{k-1}$ are true for some $k \in \mathbb{N}$. We show, that $(A1)_k$, $(A2)_k$, $(A3)_k$ and $(A4)_k$ are true as well.

First let us show that $\tilde{x}_k, x_k \in S$. By $(A3)_{k-1}$ we know that $S_{k-1} \subset S$. Then

$$\|x_k - x_{k-1}\| = \zeta_{k-1} \text{ because of (12)}$$

so that $x_k \in S_{k-1} \subset S$. Furthermore

$$\begin{aligned} \|\tilde{x}_k - x_{k-1}\| &\leqslant \|\tilde{x}_k - x_k\| + \|x_k - x_{k-1}\| \\ &\leqslant \lambda_k \|F(x_k)\| + \|x_k - x_{k-1}\| \qquad \text{by (12)} \\ &\leqslant 2\|x_k - x_{k-1}\| = 2\zeta_{k-1} \end{aligned}$$

and $x_n \in S_{k-1} \subset S$ as well.

Now let E be the identity-map in X and define the operator $U_k : X \to X$ by

(14) $$U_k := E - J_{k-1}^{-1} J_k = J_{k-1}^{-1}(J_{k-1} - J_k).$$

Then

$$\begin{aligned} \|U_k\| &\leqslant \|J_{k-1}\| \left[\|J_{k-1} - F'(x_{k-1})\| + \|F'(x_{k-1}) - J_k\|\right] \\ &\leqslant \beta_{k-1}[C_1 \|\tilde{x}_{k-1} - x_{k-1}\| + C_1 \|x_k - x_{k-1}\| + C_2 \zeta_{k-1}] \quad (A1)_{k-1}\ (7),\ (13) \\ &\leqslant 2\beta_{k-1} K \zeta_{k-1} = 2\eta_{k-1} \leqslant \frac{1}{2} \ (V3),\ (A3)_{k-1},\ (A2)_{k-1}\ (A4)_{k-1} \end{aligned}$$

By Banach's lemma (see [9], page 154/5) $E - U_k = J_{k-1}^{-1} J_k$ is invertible, so that J_k^{-1} exists as well and

(15) $$\|J_k^{-1}\| \leqslant \frac{\beta_{k-1}}{1 - 2\eta_{k-1}} = \beta_k$$

and $(A1)_k$ is shown.

From $(A2)_{k-2}$ and (15) it follows, that

(16) $$\beta_k \leqslant 2\beta_{k-1} \leqslant \ldots \leqslant 2^k \beta.$$

Next we estimate

$$\begin{aligned} \|F(x_k)\| &= \|F(x_k) - F(x_{k-1}) - J_{k-1} C_{k-1}\| \quad \text{by (3)} \\ &\leqslant \|F(x_k) - F(x_{k-1}) - F'(x_{k-1})(x_k - x_{k-1})\| \\ &\quad + \|F'(x_{k-1}) - J_{k-1}\| \zeta_{k-1} \end{aligned}$$

$$\begin{aligned} &\leqslant \frac{K}{2}[\zeta_{k-1} + \|\tilde{x}_{k-1} - x_{k-1}\|]\, \zeta_{k-1} \qquad \text{by (11), (7)} \\ &\leqslant K\zeta_{k-1}^2 \qquad \text{by } (A4)_{k-1} \end{aligned}$$

so that

$$\zeta_k \leqslant \beta_k \|F(x_k)\| \leqslant \beta_k K \zeta_{k-1}^2$$

$$\leqslant \frac{\beta_{k-1} K \zeta_{k-1}}{1 - 2\eta_{k-1}} \zeta_{k-1} \quad \text{by (12)}$$

(17) $$= \frac{\eta_{k-1}}{1 - 2\eta_{k-1}} \zeta_{k-1} \quad \text{by (12) and } (A2)_{k-1}$$

$$\leqslant \frac{1}{2} \zeta_{k-1}.$$

Therefore we have by (17)

$$\eta_k = \beta_k K \zeta_k \leqslant 2\, \beta_{k-1} K \frac{1}{2} \zeta_{k-1} = \eta_{k-1} \leqslant \frac{1}{4}$$

so that $(A2)_k$ is shown.

Let $x \in S_k$, then

$$\begin{aligned} \|x - x_{k-1}\| &\leqslant \|x - x_k\| + \|x_{k-1}\| \leqslant 2\zeta_k + \zeta_{k-1} \\ &\leqslant 2\zeta_{k-1} \quad \text{by (17)} \end{aligned}$$

and $x \in S_{k-1}$, that means $S_k \subset S_{k-1} \subset S$, that is $(A3)_k$.

At least

$$\|\tilde{x}_k - x_k\| = \lambda_k \|F(x_k)\| \leqslant \lambda_k \|J_k\| \zeta_k \quad \text{by (3)}$$

but
$$\begin{aligned} \|J_k\| &\leqslant \|J_{k-1}\| + \|J_{k-1} - J_k\| \\ &\leqslant \|J_{k-1}\| + \|J_{k-1}\| \|U_k\| \quad \text{by (14)} \\ &\leqslant \frac{3}{2} \|J_{k-1}\| \end{aligned}$$

so that by (12)

$$\|\tilde{x}_k - x_k\| \leqslant \zeta_k \quad \text{by (12)}$$

and $(A4)_k$ and even $(B1)$ is shown.

$(B2)$ From $\zeta_n \leqslant \frac{1}{2}\zeta_{n-1}$ for all $n \in \mathbb{N}$ it follows by induction that

(18) $$\zeta_n \leqslant \left(\frac{1}{2}\right)^n \zeta_0$$

and $\{\zeta_n\}_{n\in\mathbb{N}_0}$ converges to zeros. Then we have for $n, m \in \mathbb{N}_0$

$$\|x_{n+m} - x_n\| \leq \sum_{j=0}^{m-1} \|x_{n+j+1} - x_{n+j}\| = \sum_{j=0}^{m-1} \zeta_{n+j}$$

$$\leq \zeta_n \sum_{j=0}^{m-1} \left(\frac{1}{2}\right) \leq 2\,\zeta_n$$

from which it follows that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy-sequence in the closed ball S and therefore has a limit $x^* \in S$. For $m \to \infty$ we have

(19) $$\|x^* - x_n\| \leq 2\,\zeta_n$$

and from the continuity of F we conclude from the derivation of (17) that

$$0 \leq \|F(x^*)\| = \lim_{n\to\infty} \|F(x_n)\| \leq K \lim_{n\to\infty} \zeta_n^2 = 0$$

and thus x^* is a solution of (1).
Now we estimate

$$\|x^* - x_n\| \leq \|x^* - x_{n-1} + J_{n-1}^{-1} F(x_{n-1})\| \leq$$

$$\leq \|J_{n-1}^{-1}\| \{\|J_{n-1}(x^* - x_{n-1}) - F'(x_{n-1})(x^* - x_{n-1})\|\} \leq$$

$$\leq \|J_{n-1}^{-1}\| \frac{K}{2} \{\|\tilde{x}_{n-1} - x_{n-1}\| + \|x_{n-1} - x^*\|\} \|x_{n-1} - x^*\|$$

by $(V3)$, (7), (11)

and

$$\|\tilde{x}_{n-1} - x_{n-1}\| = \lambda_{n-1} \|F(x^*) - F(x_{n-1})\| \leq \lambda_{n-1} \left\{ \|F'(x^*)\| + \frac{K}{2}\|x^* - x_{n-1}\| \right\} \|x^* - x_{n-1}\|$$

so that

(20) $$\|x^* - x_n\| \leq M_{n-1} \|x^* - x_{n-1}\|^2$$

with

$$M_n = \|J_n^{-1}\| \frac{K}{2} \left\{ \lambda_n \left(\|F'(x^*)\| + \frac{K}{2}\|x^* - x_n\| \right) + 1 \right\}, \quad n \in \mathbb{N}_0.$$

Since $\|F'(x^*)\|$ is constant and we know from (18) and (19) that

$$\|x^* - x_n\| \leq 2\zeta_n \leq 2\zeta_0$$

and from (12)

$$\lambda_n \leq \frac{2}{3\|J_{n-1}\|} \leq \frac{2}{3}\|J_{n-1}^{-1}\|$$

we only have to estimate $\|J_n^{-1}\|$ uniformly in n. Defining $T_n := E - J_0^{-1} J_n$ we have as in the proof of $(A1)_n$ for $n \in \mathbb{N}$

$$\|T_n\| \leq \beta[2C_1\zeta + C_1\|x_0 - \tilde{x}_n\| + C_2\|x_0 - x_n\|]$$

$$\leq \beta[2C_1\zeta + (C_1 + C_2)\|x_0 - x_n\| + C_1\|\tilde{x}_n - x_n\|]$$

$$\leq \beta[2C_1\zeta + (C_1 + C_2)2\zeta + C_1\zeta_n] \quad \text{by } (A3)_n$$

$$\leq \frac{7}{2}\beta K\zeta = \frac{7}{2}\eta < 1 \quad \text{by } (V1), (V3), (18)$$

and

$$\|J_n^{-1}\| \leq \frac{\beta}{1 - \frac{7}{2}\eta},$$ which is also true for $n = 0$ because of $(V2)$.

Therefore

$$\forall N \in \mathbb{N}_0 : M_n \leq \frac{\beta K}{1 - \frac{7}{2}\eta} \cdot \frac{1}{2} \left\{ 1 + \frac{\beta}{1 - \frac{7}{2}\eta} (\|F'(x^*)\| + K\zeta_0) \right\} =: C$$

and from (20) the quadratic convergence follows

$$\forall n \in \mathbb{N} : \|x^* - x_n\| \leq C\,\|x^* - x_{n-1}\|^2$$

and $(B2)$ is shown.
$(B3)$ From (18) and (19) we have

$$\forall n \in \mathbb{N}_0 : \|x^* - x_n\| \leq \left(\frac{1}{2}\right)^{n-1} \zeta$$

that is the a-priori bound 1.
From the deviation of (17) we have

(21) $$\zeta_n < \frac{\eta_{n-1}}{1 - 2\eta_{n-1}}\,\zeta_{n-1}$$

so that by (12) and $(A2)_n$

(22) $$\eta_n < \frac{\eta_{n-1}}{1 - 2\eta_{n-1}} \cdot \frac{\beta_{n-1} K}{1 - 2\eta_{n-1}}\,\zeta_{n-1} < \frac{\eta_{n-1}^2}{(1 - 2\eta_{n-1})^2}.$$

Now let us define the function

$$N(h) := \frac{1 - \sqrt{1 - 4h}}{2h} = \frac{2}{1 + \sqrt{1 - 4h}} \quad \text{for } 0 < h \leq \frac{1}{4}$$

then

$$(23) \qquad 1 < N(h) \leqslant 2 \text{ and}$$

$$\zeta_{n+1}N(\eta_{n+1}) \leqslant \frac{\eta_n}{1-2\eta_n}\,\zeta_n\,\frac{2}{1+\sqrt{1-4\eta_{n+1}}} \quad \text{by (21)}$$

$$\leqslant \frac{2\eta_n}{1-2\eta_n}\,\zeta_n\,\frac{1-2\eta_n}{1-2\eta_n+\sqrt{1-4\eta_n}} \quad \text{by (22)}$$

$$\leqslant \frac{2\eta_n}{1-2\eta_n+\sqrt{1-4\eta_n}}\,\zeta_n = \frac{1-2\eta_n-\sqrt{1-4\eta_n}}{2\eta_n}\,\zeta_n = \zeta_n N(\eta_n) - \zeta_n$$

from which it follows that

$$\|x_{n+m} - x_n\| \leqslant \sum_{j=0}^{m-1} \zeta_{n+j} \leqslant \zeta_n N(\eta_n) - \zeta_{n+m}N(\eta_{n+m})$$

and for $m \to \infty$ because of (23) and (18)

$$(24) \qquad \|x^* - x_n\| \leqslant \zeta_n N(\zeta_n) = \frac{2\zeta_n}{1+\sqrt{1-4\eta_n}}.$$

For $n \geqslant 1$ we then have by (17) and (22)

$$\|x^* - x_n\| \leqslant \frac{2\eta_{n-1}\zeta_{n-1}}{1-2\eta_{n-1}+\sqrt{1-4\eta_{n-1}}} = 2\,\frac{\beta_{n-1}K\zeta_{n-1}^2}{1-2\eta_{n-1}+\sqrt{1-4\eta_{n-1}}}$$

From (21) and (22) we have

$$\zeta_n \leqslant 2\eta_{n-1}\zeta_{n-1}$$

$$\eta_n \leqslant 4\eta_{n-1}^2.$$

Now we show by induction

$$(25) \qquad \eta_n \leqslant \frac{(4\eta)^{2^n}}{4} \text{ and } \zeta_n \leqslant \frac{(4\eta)^{2^n-1}}{2^n}\,\zeta$$

$$n = 0 : \eta_0 = \eta = \frac{(4\eta)^{2^0}}{4} \text{ and } \zeta_0 = \zeta = \frac{(4\eta)^{2^0-1}}{2^0}\,\zeta.$$

Now let (25) be true for some $k \in \mathbb{N}_0$, then

$$\eta_{k+1} \leqslant 4\eta_k^2 \leqslant 4\left(\frac{(4\eta)^{2^k}}{4}\right) = \frac{(4\eta)^{2^{k+1}}}{4}$$

$$\zeta_{k+1} \leqslant 2\eta_k\zeta_k \leqslant \frac{1}{2}(4\eta)^{2^k}\frac{(4\eta)^{2^k-1}}{2^k}\,\zeta = \frac{(4\eta)^{2^{k+1}-1}}{2^{k+1}}\,\zeta$$

so that (25) is proved for all $n \in \mathbb{N}_0$. Thus

$$\|x_{n+m} - x_n\| \leqslant \sum_{j=0}^{m-1}\zeta_{n+j} \leqslant \frac{(4\eta)^{2^n-1}}{2^n}\,\zeta\left(1+\frac{1}{2}+\frac{1}{4}+\dots\right) \leqslant \frac{(4\eta)^{2^n-1}}{2^{n-2}}\,\zeta$$

from which the a-priori bound 2 follows as $m \to \infty$. Therefore $(B3)$ is shown.

$(B4)$ Let x' be a solution of (1) with $x^* \neq x' \in S$. Then

$$\|x' - x_0\| \leqslant 2\zeta.$$

We show by induction

$$(26)_n \qquad \forall n \in \mathbb{N}_0 : \|x' - x_n\| \leqslant \left(\frac{1}{2}\right)^{n-1}\zeta.$$

For $k \geqslant 1$ we have analogously to the deviation of (20)

$$\|x^* - x_k\| \leqslant \frac{\beta_{k-1}K}{2}\left[\|\tilde{x}_{k-1} - x_{k-1}\| + \|x_{k-1} - x'\|\right]\cdot\|x_{k-1} - x'\| \leqslant$$

$$\leqslant \frac{1}{2}2^{k-1}\beta K\left[\left(\frac{1}{2}\right)^{k-2}\zeta + \left(\frac{1}{2}\right)^{k-2}\zeta\right]\left(\frac{1}{2}\right)^{k-2}\zeta$$

by $(26)_{k-1}$, (18), (16) and $(A4)_{k-1}$.

$$= 2\beta K\zeta\left(\frac{1}{2}\right)^{k-2}\zeta = 2\eta\left(\frac{1}{2}\right)^{k-2}\zeta \leqslant \left(\frac{1}{2}\right)^{k-1}\zeta.$$

Therefore x_n tends to x' as $n \to \infty$ in contradiction to $x' \neq x^*$. Thus x^* is the only solution of (1) in S and (13) follows from (24) for $n = 0$.

Q.E.D.

REMARK 1. The a-posteriori bound can be improved if we take $\|J_{n-1}^{-1}\|$ instead of β_{n-1} and if K-respectively C_1 and C_2 — is restricted at each iteration step to the ball S_n.

EXAMPLE 1. This example shows the optimality of the a-posteriori bound.

Let $X = \mathbb{R}$, $F(x) = x^2 - \frac{104}{25}x - \frac{22}{25}$, so that $F(x) = 0$ has the solutions

$$x^* \doteq -0.2017536$$

$$\text{and } y^* \doteq 4.3617536$$

with $x_0 = 0$, $\tilde{x}_0 = 0.1$ and $J(x, y) = x + y - \frac{104}{25}$ we have $K = 2$, $\beta \doteq 0.2463055$, $\zeta \doteq 0.216749$ and therefore $\eta \doteq 0.1067728 < \frac{1}{4}$, $x_1 = -0.2167487$ so that the exact error $\|x_1 - x^*\| = 1.49951 * 10^{-2}$ while the a-posteriori bound is $\|x_1 - x^*\| \leqslant 1.49951 * 10^{-2}$ which is the same value.

REMARK 2. If we choose $\lambda_n = 1$ for all $n \in \mathbb{N}$, that means if we take (2) instead of (5) even for $n = 0$, we have the original method of Steffensen. Then we get the following

COROLLARY. *Let X, D, F, J be given as in the preceding theorem. If there exists an element $x_0 \in D$, such that*

(V4) $J(\tilde{x}_0, x_0)^{-1}$ *exists with* $\tilde{x}_0$ *from* (2)

$$S := \{x \in X \mid \|x - x_0\| \leqslant r\} \subset D \text{ where}$$

$$r \geqslant \max\{\|F(x_0)\|,\ 2\|J(\tilde{x}_0, x_0)^{-1}F(x_0)\|\}$$

(V2) $\|J(x_0, x_0)^{-1}\| \leqslant \beta$

(V5) $\eta := \beta K(\zeta + r) < \frac{1}{2}$ *with* $\zeta \geqslant \|J(x_0, x_0)^{-1}F(x_0)\|$

and $K = \max\{C_1 + C_2, 2C_1\}$ *with* C_1, C_2 *from* (7) *where,* $x, y \in S$ *then*
(B5) *the method of Steffensen determinated by* (2), (3), (4) *gives a sequence* $\{x_n\}_{n \in \mathbb{N}} \subset S$, *which*
(B6) *converges quadratically against a solution* $x^* \in S$ *for* (1).
(B7) *We have the following error bounds*

$$\forall n \in \mathbb{N}_0 : \|x_n - x^*\| \leqslant \left(\frac{1}{2}\right)^{n-1} \zeta \text{ (a-priori bound 1)}$$

$$\forall n \in \mathbb{N}_0 : \|x_n - x^*\| \leqslant \frac{(2\eta)^{2^n - 1}}{2^{n-1}} \zeta \text{ (a-priori bound 2)}$$

$$\forall n \in \mathbb{N} : \|x_n - x^*\| \leqslant \frac{\beta_{n-1} K(\zeta_{n-1} + \tilde{\zeta}_{n-1})}{1 - \eta_{n-1} + \sqrt{1 - 2\eta_{n-1}}} \zeta_{n-1}$$

(a-posteriori bound)

with

$$\eta_0 := \eta,\quad \beta_0 := \beta,\quad \zeta_0 := \zeta,\quad r_0 := r$$

$$\zeta_n := \|x_{n+1} - x_n\|,\quad \tilde{\zeta}_n := \|\tilde{x}_n - x_n\| = \|F(x_n)\|$$

$$r_n := \max\{2\zeta_n, \tilde{\zeta}_n\}, \beta_n := \frac{\beta_{n-1}}{1 - \eta_{n-1}}$$

$$\eta_n = \beta_n K(\zeta_n + r_n)$$

(B8) x^* *is the only solution of* (1) *in* S *and*

$$\|x^* - x_0\| \leqslant \frac{1 - \sqrt{1 - 2\eta}}{\eta} \zeta. \tag{27}$$

The corollary can be proved in a similar way as the theorem, so that the proof can be omitted. We only give an example which shows that our error-bounds are much better than the bounds from [1], [17], [10] and [7].

$$F(x) = \begin{pmatrix} 5 & - \xi_1\xi_2 \\ \xi_1 & \xi_1\xi_2 + \xi_2 - 13 \end{pmatrix} \doteq 0,\ x = (\xi_1, \xi_2)^T \tag{28}$$

be given. It is easy to see that a solution of this system, which is symmetric in ξ_1 and ξ_2, is given by the roots of the quadratic polynomial $t^2 - 8t + 5$, which are $4 \pm \sqrt{11}$, that means

$$x^* = \begin{pmatrix} 4 + \sqrt{11} \\ 4 - \sqrt{11} \end{pmatrix} \doteq \begin{pmatrix} 7.3166 & 24790 & 35540 & 0 \\ P.6833 & 75209 & 65560 & 0 \end{pmatrix}$$

is a solution of (29). If we choose for $J(\cdot, \cdot)$ the symmetric operator

$$\begin{pmatrix} -\frac{\xi_2 + \xi_2}{2} & -\frac{\xi_1 + \eta_1}{2} \\ 1 + \frac{\xi_2 + \eta_2}{2} & 1 + \frac{\xi_1 + \eta_1}{2} \end{pmatrix} \quad \begin{matrix} x = (\xi_1, \xi_2)^T \\ y = (\eta_1, \eta_2)^T \end{matrix}$$

then the example is simple enough to compare all known bounds except the estimates of Chen [5], where a contraction-condition has to be fulfilled. With

$$x_0 = (7.317,\ 0.683)^T$$

using the $\|\cdot\|_\infty$-norm we have $\beta \doteq 2.3572$, $K = 2$, $\zeta \doteq 3.7533 \cdot 10^{-4}$, $\tilde{\zeta} = 2.489 \cdot 10^{-3}$ and $\eta = 0.01351 < \frac{1}{2}$ and Steffensen's method gives

$$\begin{aligned} x_0 &= 7.317 && 0.683 \\ x_1 &= 7.31662\ 46707\ 57539 && 0.68337\ 53292\ 42461 \\ x_2 &= 7.31662\ 47903\ 55388 && 0.68337\ 52096\ 44612 \\ x_3 &= 7.31662\ 47903\ 55400 && 0.68337\ 52096\ 44600. \end{aligned}$$

The error-estimates are as follows

	$\|x^* - x_0\|_\infty$	$\|x^* - x_1\|_\infty$	$\|x^* - x_2\|_\infty$
Balász [1]	not applicable	$534146.7 \cdot 10^{-7}$	$9078077500 \cdot 10^{-13}$
Ul'm [19]	$40.1 \cdot 10^{-4}$	$2019.0 \cdot 10^{-7}$	$5630000 \cdot 10^{-13}$
Koppel' [9]	$39.6 \cdot 10^{-4}$	$2033.3 \cdot 10^{-7}$	$6300000 \cdot 10^{-13}$
Johnson & Scholz [7]	$37.9 \cdot 10^{-4}$	$1413.3 \cdot 10^{-7}$	$1010000 \cdot 10^{-13}$
a-priori 2 aus (B7)	$7.51 \cdot 10^{-4}$	$101.4 \cdot 10^{-7}$	$36966 \cdot 10^{-13}$
(28)	$3.78 \cdot 10^{-4}$	—	—
a-posteriori aus (B7)	—	$25.7 \cdot 10^{-7}$	$2.6 \cdot 10^{-13}$
exact error	$3.76 \cdot 10^{-4}$	$1.2 \cdot 10^{-7}$	$0.122 \cdot 10^{-13}$

and we see, that especially for $n \geqslant 1$, the only error-bound of practical use is given by the a-posteriori bound in (B7). At least it should be mentioned that for many problems (for example (28) with $x_0 = (7.31, 0.683)$) only the corollary of this paper is applicable.

REFERENCES

[1] Balász, M.: *A note on the convergence of Steffensen's method*, Math., Revue d'Anal. Num. Th. Approx., **10**, 5–10 (1981).

[2] Balász, M. and Goldner, G.: *On Steffensen's method in Fréchet spaces*, Stud. Univ. Babes-Bolyai Math., **28**, 34–37, (1983).

[3] Baptist, P.: *Konvergenz and monotone Einschließung für das Steffensen-Verfahren*, Elemente der Mathematik, **37**, 33–40 (1982).

[4] Bel'tyukov, B. A.: *A method of solving nonlinear functional equations*, USSR Comp. Math. Phys., **5**, 210–217 (1965).

[5] Chen, K. W.: *Generalisation of Steffensen's method for operator equations*, Comment. Math. Univ. Carolinae, **5**, 47–77 (1964).

[6] Döring, B.: *Über das Newtonsche Näherungsverfahren*, Math.-Phys. Sem.-Ber., **16**, 27–40 (1969).

[7] Hofmann, W.: *Monotonieeigenschaften des Steffensen-Verfahrens*, Aequationes Math., **12**, 21–31 (1975).

[8] Johnson, L. W. and Scholz, D. R.: *On Steffensen's method*, SIAM J. Num. Anal., **5**, 296–302 (1968).

[9] Kantorovich, L. V. and Akilov, G. P.: *Functional analysis*, 2nd ed., New York, Pergamon Press (1982).

[10] Koppel', H.: *Convergence of the generalised method of Steffensen* (in Russian) Eesti NSV Tead. Akad. Toimetised Füüs. Mat. Techn.-tead. Seer., **15**, 531–539 (1966).

[11] Mönch, W.: *Inversionsfreie Verfahren zur Einschließung von Nullstellen nichtlinearer Operatoren*, Beitr. Numer. Math., **2**, 125–136 (1974).

[12] Ortega, J. M. and Rheinboldt, W. C.: *Iterative solutions of nonlinear equations in several variables*, 1st ed., New York, Akad. Press (1970).

[13] Schmidt, J. W.: *Konvergenzgeschwindigkeit der Regula-falsi und des Steffensen-Verfahrens in Banachräumen*, ZAMM, **46**, 146–148 (1966).

[14] Schmidt, J. W.: *Regula-falsi-Verfahren mit konsistenter Steigung und Majorantenprinzip*, Period. Math. Hung., **5**, 187–193 (1974).

[15] Schneider, N.: *Results about monotone convergence of Steffensen-like-methods*, BIT, **21**, 347–354 (1981).

[16] Steffensen, J. F.: *Remarks on iteration*, Skand. Aktuar. Tidskv., **16**, 64–72 (1933).

[17] Ul'm, S. Y.: *Extension of Steffensen's method for solving nonlinear operator equations*, USSR Comp. Math., **4**, 159–165 (1964).

Received 24.II.1985

Mathematisches Institut der Universität Düsseldorf
Universitätsstraße 1
D–4000 Düsseldorf 1