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# FUNCTIONAL, NORMAL WRITING LANGUAGE. PROOF OPERATORS

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In this paper I propose a new type of formal language, with a new structure, adapted for mathematical thought. Being created as a model for usual mathematical language, this one is characterized by a linear, normal, free of brackets, computerizable, Lukasiewicz writing. These features allow an improvement of the mathematical process by means of proof operators.

The introduction of the new language will be done with the aid of a personal axiomatic theory, of geometrical essence, which I call "the K-spaces theory".

The paper comprises three parts

1. The functional, normal writing language
2. Proof operators
3. K-spaces theory

## 1. THE FUNCTIONAL, NORMAL WRITING LANGUAGE

### 1.1. *The signs of the language*

logical symbols *A* existential quantifier
*B* disjunction
*C* implication
*D* definition
*I* quantifier "there exists only one"
*M* equivalence
*N* negation
*X* conjunction
*Z* universal quantifier

constant predicates *E*, *F*, *G*, *H*, *K*, *P*, *S*, *U*, *V*
figures *O*,1 2, 3, 4, 5, 6, 7, 8, 9 and *Y*, *L*

### 1.2. *Formation rules*

—The figures and letters are called signs.
— The numbers are formed either of a single figure, or of more figures followed by $n-1$ signs "*Y*", where $n$ is the number of figures. The

figures succeed in reverse order in comparison to the usual writing. For example, number 12 becomes $21Y$, number 127 becomes $721YY$, and so on.

— The signs written one near the other, from the left to the right, are called sequences (when citing them, we use quotation marks).

— By joining two sequences together, we obtain another sequence.

— A sequence is included in another sequence if each sign of the first is a sign of the latter.

— A number from a sequence is an index, an order-number, a variable-number of "$F$" or "$D$", or "$O$".

— Indexes and order-numbers are natural nonzero numbers.

— A number is a variable-number of "$F$" or "$D$" iff it is followed by "$F$" or "$D$".

— A number is an order-number iff it is followed by "$L$" or "$kF$" or "$kD$", where $k$ is a number.

— Variables are sequences.

— Indexes are variables.

— If $m \in N^*$, $k \in N$ and $x_1, x_2, \ldots, x_k$ are variables, then "$x_1 x_2 \ldots x_k \; mkF$" is a variable.

— When counting the variables of a sequence, we do not take into account repetitions and variables included in other variables.

— A well-formed formula (wff) is a sequence.

—If $p$ and $q$ are wff, then "$pqB$", "$pqC$", "$pqM$", "$pqX$" are wff. In this case, $p$ is called the first member of the respective operation, and $q$ is called the second member.

— If $p$ is a wff, then "$pN$" is a wff. $p$ is the member of "$N$".

— If $x$ is a variable, then "$xG$", "$xP$", "$xS$" are wff. In this case $x$ is called the member of the respective operation.

— If $x$ and $y$ are variables then "$xyE$", "$xyH$", "$xyK$", "$xyU$", "$xyV$" are wff, $x$ being the first member of the operation and $y$ the second one.

— If $m \in N^*$, $k \in N$ and $x_1, x_2 \ldots, x_k$ are variables, then "$x_1 x_2 \ldots \ldots x_k \; mk \, D$" is a wff. Then $x_1, x_2, \ldots, x_k$ are called members, $m$ is the order-number, and $k$ is the variable-number of "$D$".

— If $p$ and $q$ are wff and if $i \in N$, then "$piPZ$" and "$piPqZ$" are wff ("$P$" can be replaced by "$S$" or "$G$" and "$Z$" by "$A$" or "$I$").

"$piPZ$" is a simple quantification and "$iPZ$" is its basis
"$piPqZ$" is a conditioned quantification and "$iPqZ$" is its basis
"$p$" is the quantifier field
"$i$" is the quantifier index
"$P$" is the quantifier domain
"$q$" is the quantifier condition
"$Z$" is the quantifier

— In *a wff* any index equal to a quantifier index is called loose index of the quantifier.

— "$O$" is a *wff*.

— Statements are *wff*.

— Statements are numbered. The order-number of the statement is preceded by $A$, $T$, or $D$ (axiom, theorem or definition).



— The order-numbers which are included in a statement must be less than the order-number of the statement.

— In a statement, the indexes are, in order of their appearance, the numbers 1, 2, ... (with possible repetitions).

— In a statement, the greatest index must be equal to the number of quantifiers.

— In a statement, two different quantifiers have different indexes.

— A *wff* is called a successive quantification if each of the quantifiers is the last sign of the field of the next quantifier or is the last sign of the formula.

— A *wff* is called a successive quantification starting with the $i$-th quantifier ($i \in N^*$) if for all $j \in N^*$, $i \leqslant j < k$, where $k$ is the number of quantifiers of the formula, the $j$-th quantifier is the last sign of the field of the $j+1$ th quantifier, and the $k$-th quantifier is the last sign of the formula.

— Let $m$ be a *wff* and $x$ one of the following signs. It is called the stretch of $x$

*a*) the field and the basis of $x$, if $x$ is "$A$" or "$I$" or "$Z$".
*b*) the first and the second member of $x$ together with $x$, if $x$ is "$B$", "$C$", "$E$", "$H$", "$K$", "$M$", "$U$", "$V$", "$X$".
*c*) the member of $x$ together with $x$, if $x$ is "$G$", "$P$", "$S$" or "$N$".
*d*) the members, the order-number, the variable-number of $x$ together with $x$, if $x$ is "$D$",

— If $p$, $q$, $r$ are *wff* then
*a*) $r$ is stated in $r$.
*b*) if $r$ is stated in $p$, then $r$ is stated in "$pqX$".
*c*) if $r$ is stated in $q$, then $r$ is stated in "$pqX$".
*d*) if $p$ is stated in $r$ and $q$ is stated in $r$, then "$pqX$" is stated in $r$.

— If "$piPZ$" and "$piPqZ$" are quantifications) "$P$" may be replaced by "$S$" or "$G$" and "$Z$" by "$I$" or "$A$") then
*e*) if $r$ is stated in $p$, then $r$ is stated in "$piPZ$".
*f*) if $r$ is stated in $p$, then $r$ is stated in "$piPqZ$".
*g*) if $r$ is stated in $q$, then $r$ is stated in "$piPqZ$".
*h*) "$iP$" is stated in "$piPZ$".
*i*) "$iP$" is stated in "$piPqZ$".

— We say that a sign is not included in any condition (*nic*) in a *wff p* if it is not included in the condition of any quantifier of $p$.

— If $p$ is a *wff*, then the negation of $p$ is made as follows

Verify that there exists no "$C$", "$I$", "$M$" *nic* in $p$.
*a* ! write $p$.
*b* ! every "$A$" which is *nic*] in $p$ turns into "$Z$" and every "$Z$" *nic* in $p$ turns into "$A$".
*c* ! every "$B$" *nic* in *b* ! turns into "$X$" and every "$X$" *nic* in *b* ! turns into "$B$".
*d* ! add "$N$" after the field of every *nic* quantifier from *c* ! whose field does not end by another quantifier or by "$B$" or "$X$".
*e* ! add "$N$" after each member, which does not end by a quantifier or by "$B$" or "$X$", of any operation "$B$" or "$X$" *nic* in *d* !.

*f*! cancel out "*NN*" any time it appears.

— Let *p* be a *wff*.

*p* is a successive quantification *nic*( *sqn*) if each quantifier *nic* in *p* is the last sign of the field of the next quantifier *nic* in *p* or is the last sign of the formula.

*p* is a *sqn* starting with the *i*-th quantifier *nic* in *p* ($i \in N^*$) if for all $j \in N^*$, $i \leqslant j < k$, where *k* is the number of quantifiers *nic* in *p*, the *j*-th quantifier *nic* in *p* is the last sign of the field of the $j+1$ th quantifier *nic* in *p*, and the *k*-th quantifier *nic* in *p* is the last sign of *p*.

1.3. *A detailed treatise of some examples of writing in the functional, normal writing language.*

The definition 7 from the *K*-spaces theory may be expressed as follows.

"For all, *A*, *B* 'planes', *A* is point-included in *B* if and only if for each point *x* belonging to *A*, *x* belongs to *B*".

We note the variables *A*, *B*, *x* by "1", "2", "3". Then "*A* is point-included in *B*" will be written "1272*D*", where the first two signs represent the varibles *A* and *B*, "7" is the order-number of the definition, the second "2" indicates that we have a two-variables predicate, and "*D*" is the definition symbol. Therefore

"*A* is point-included in *B*" becomes "1272*D*".

"*x* belongs to *B*" becomes "32*H*".

"For each point *x* belonging to *A*, *x* belongs to *B*" becomes "32*H*3*P*31*HZ*".

"*A* is point-included in *B* if and only if for each point *x* belonging to A, *x* belongs to *B*" becomes "1272*D*32*H*3*P*31*HZM*".

"For all *A*, *B* 'planes', *A* is point-included in *B* if and only if for each point *x* belonging to *A*, *x* belongs to *B*" becomes "1272*D*32*H*3*P*31*HZM*2-*GZ*1*G*$_Z$", which is just the new form of the definiton 7.

The axiom 2 of the *K*-spaces theory is usually formulated as follows:

"For each straight line *d* there exists only one point *x* such that *x* is the beginning of *d*".

We note the variables *d* and *x* by "1" and "2". Therefore

"*x* is the beginning of *d*" becomes "21 *U*".

"There exists only one point *x* such that *x* is the beginning of *d*" becomes "21 *U*2*PI*".

"For each straight line *d* there exists only one point *x* such that *x* is the beginning of *d*" becomes "21 *U*2*PI*1*SZ*" and, rearranging the indexes of the variables according to the conditions imposed, the axiom 2 becomes "12 *U*1*PI*2*SZ*".

## 2. PROOF OPERATORS

The task of this chapter is only to emphasize the qualities of our language. For this reason the list of usual proof operators remains open and the various operators may suffer transformations. Each proof operator is an algorithm with the aid of which we can act on the statements.



In an algorithm, the operations on the statements have been noted by *a*, *b*, *c*,. . . ,followed by "." or "!", where "." means that something is added to the previous result, and "!" means that the previous result or another indicated one is taken again and modified.

2.1. *Operator F*

*mF* expresses the function that results from *m*.

Verify that

— there exists $i \in N^*$ such that the *i*-th quantifier of *m* is "*I*" and *m* is a successive quantification starting with the *i*-th quantifier.

— if *k* is the number of quantifiers of *m*, then for all *j*, $i < j \leqslant k$, the *j*-th quantifier of *m* is "*Z*".

*a*. write *m* till to the field of the *i*-th quantifier exclusively.

*b*. write the index and the domain of the *i*-th quantifier of *m*.

*c*. if the *i*-th quantifier of *m* has a condition, then write the condition followed by "*X*", if not, go to *d*.

*d*. write the field of the *i*-th quantifier of *m*.

*e*. write "*X*".

*f*. write what is left in *m* after the *i*-th quantifier.

*g*. replace in *f* all the indexes equal to the index of the *i*-th quantifier of *m* by "*mOF*" if $i=k$, otherwise replace them by "$j_{i+1}\, j_{i+2} \ldots j_k\, m(k-1)F$", where $j_l$ is the index of the *l*-th quantifier of *m*.

*h*! order the indexes.

2.2. *Operator X*

*mnX* infer from *m* and *n* their conjunction.

*a*. write *m*.

*b*. write *n* increasing its indexes with the number of quantifiers of *m*.

*c*. write "*X*".

2.3. *Operator 1T*

*m*1*T* extracts the first member of an "*X*".

Verify that *m* ends by "*X*" or that there exists $i \in N$ such that *m* is a *sqn* starting with the *i*-th quantifier *nic* in *m* and that the field of this quantifier ends by "*X*".

*a*. write the first member of that "*X*".

*b*. write what is left in *m* after that "*X*".

*c*! order the indexes.

2.4. *Operator E*

*miE*($i \in N^*$) applies the symmetry to the *i*-th operation "*E*" or "*M*" from *m*.

Verify that *m* contains at least *i* operations "*E*" or "*M*".

*a*. write *m* till to the first member (exclusively) of the *i*-th operation "*E*" or "*M*".

*b*. write the second member of the $i$-th operation "$E$" or "$M$" of $m$.

*c*. write the first member of the $i$-th operation "$E$" or "$M$" of $m$.

*d*. write the $i$-th operation "$E$" or "$M$" of $m$.

*e*. write what is left in $m$ after the $i$-th operation "$E$" or "$M$" of $m$.

*f* ! order the indexes.

2.5. *Operator M*

$mnii_1 \ldots i_kkM$ ($i \in \mathbf{N}^*$, $k \in \mathbf{N}$, $i_j \in \mathbf{N}$) replaces in $m$ the stretch of the $i$-th sign by an equivalent formula that results from $m$ when replacing the loose indexes of the $l-k+j$ th quantifier of $n$ by the $i_j$-th variable from the stretch of the $i$-th sign of $m$ (we do not count the variables which are loose indexes of the quantifiers from the stretch of the $i$-th sign of $m$), where $l$ is the number of quantifiers of $n$.

Verify that

— the $i$-th sign of $m$ is "$A$", "$B$", "$C$", "$D$", "$E$", "$G$", "$H$", "$I$", "$K$", "$M$", "$N$", "$P$", "$S$", "$U$", "$V$", "$X$", "$Z$".

— if $k \neq 0$, the stretch of the $i$-th sign of $m$ contains at least max $(i_1, i_2, \ldots, i_k)$ variables which are not loose indexes of the quantifiers from the stretch of the $i$-th sign of $m$.

— if $k \neq 0$, the last $k$ quantifiers of $n$ are "$Z$" and $n$ is a successive quantification starting with the $l-k+1$ th quantifier and the field of the $l-k+1$th quantifier of $n$ ends by "$M$".

— if $k = 0$, $n$ ends by "$M$".

— the stretch of the $i$-th sign of $m$ and the first member of the operation "$M$" (indicated above) of $n$ have the same number of quantifiers (we note this number by $p$).

*a* ! write $n$ increasing each index of it with the number of quantifiers of $m$.

*b* ! for all $q \in \{1, 2, \ldots, p\}$, replace the loose indexes of the $q$-th quantifier from the first member of the operation "$M$" from *a* ! by the index of the $q$-th quantifier from the stretch of the $i$-th sign of $m$.

*c* ! for all $j \in \{1, 2, \ldots, k\}$, replace the loose indexes of the $1-k+j$ th quantifier from *b* ! by the $i_j$-th variable from the stretch of the $i$-th sign of $m$ (we do not count the variables that are loose indexes of the quantifiers from the stretch of the $i$-th sign of $m$). Do not write the basis of this quantifier any more, verify that the result of the substitution accomplished in the basis (without "$Z$") represents one or two formulae (depending on the type of quantification, simple or conditioned) which are stated in $m$.

Verify that the first member of "$M$" from *c* ! coincides with the stretch of the $i$-th sign of $m$.

d write $m$ replacing the stretch of the $i$-th sign of $m$ by the second member of "$M$" from *c* !

*e* ! order the indexes.



2.6. *Operator* 1*Q*

$mi1Q$ ($i \in \mathbf{N}^*$) suppresses the condition of the $i$-th quantifier of $m$.

Verify that

— $m$ contains at least $i$ quantifiers.

— the $i$-th quantifier from $m$ has a condition.

*a*. write $m$ till to the field of the $i$-th quantifier exclusively.

*b*. write the condition and the field of the $i$-th quantifier of $m$.

*c*. if the $i$-th quantifier of $m$ is "$A$" or "$I$", write "$X$", and if it is "$Z$", write "$C$".

*d*. write the index, the domain and the $i$-th quantifier of $m$.

*e*. write what is left in $m$ after the $i$-th quantifier.

*f* ! order the indexes.

2.7. *Operator N*

$mN$ supposes the contrary of $m$.

Verify that there exists no "$C$", "$I$", "$M$" *nic* in $m$.

*a*. memorize $m$.

*b* ! write the negation of $m$.

2.8. *Operator c*

$mC$ recognizes the contradiction of $m$.

Verify that

— $m$ ends by "$X$" or there exists $i \in \mathbf{N}^*$ such that $m$ is *sqn* starting with the $i$-th quantifier *nic* in $m$, being formed only of quantifiers "$A$" and the field of the $i$-th quantifier *nic* in $m$ ends by "$X$".

— if we note by $n$ the first member of the operation "$X$" indicated above, there exists no "$C$", "$I$", "$M$" *nic* in $n$.

— by making the negation of $n$ followed by increasing the indexes with the number of quantifiers of $n$, we obtain the second member of the operation "$X$" indicated above.

*a*. write what is in the memory (from a previous $N$).

## 3. K-SPACES THEORY

From now on we shall abbreviate the words axiom, theorem, definition by *A*., *T*., *D*. written at the beginning of the line and followed by the order-number of the respective statement. On the same line, after a blank space, we write the statement, which may be continued, if necessary, on the following line, leaving four blank spaces. After a theorem follows the proof, written from the beginning of a new line.

We must permanently take care not to confound the signs of the language, which appear only in statements, with the notations of the proof operators, which appear in the proofs of the theorems.

3.1. *The list of statements*

A.1 12H2G32KZ3S13UZ1PZ
A.2 12U1PI2SZ
T.3 121F2H2G12KZ1SZ
3U2F21S13213S2T
A.4 12E2S121F221FE12L1F22L1FEXZ1SZ
A.5 12U32VX2SA3PZ1PZ
T.6 12U32VX2SI3PZ1PZ
D.7 1272D32H3P31HZM2GZ1GZ
D.8 1282D32K3S31KZM2GZ1GZ
D.9 1292D1272D1282DXM2GZ1GZ
A.01Y 12E2G1292D2192DXZ1GZ
D.11Y 111Y1D23V24UX4S41KI3S31KI2P21HZM1GZ
A.21Y 12HN1PZ2GI
T.31Y 121YOFKN1SZ
31YN2F1T11S21YF1TX2K3212S21YF2T31S42V2TC
T.41Y 12H1P121YOFHZ2GZ
21YF2T41YN1Q15V22V1T24KXC
T.51Y 21YOF172D1GZ
41Y21YF1TX71E31Y122M1T
T.61Y 12K1S121YOFKZ2GZ
31Y61YN1Q15V22V1T24KXC
T.71Y 21YOF182D1GZ
61Y21YF1TX81E31Y122M1T
T.81Y 21YOF192D1GZ
51Y71YX1R21YF1TX91E91Y212M1T
T.91Y 12V13UX3S321YOFKI2S221YOFKI1P121YOFHZ
21YF2T91Y13K23K1I4IN7Q1TXC6J3J12K12K
T.02Y 21YOF11Y1D
91Y21YF1TX11Y1E73Y11M1T
T.12Y 121YOF82D1G121YOF72DZ
12Y21YF1TX89212M1TN21Q12K1T21YF1TX79212M1T32F1T31S342
S31Z1T11Q2T24K2F
1T21YF2T11S2TXC21YF1TX81E31Y122M1T
T.22Y 121YOF92D1G121YOF2DZ
12Y921YF1TX22K12K31Z1T11G2T12Q11S2T
T.32Y 121YOF72D121YOF92DM1GZ
22Y11Q921YF1TX22K12K31Z1T11G1T12Q1T11QX1R12G
T.42Y 1172D1GZ
42Y75112MN11QC71E9112M

It goes without saying that the list of statements of the *K*-spaces theory does not end here. However, taking into consideration that the purpose of this paper is to present our language, 1 consider the list to be wide enough.



3.2. *Examples of proofs*

The proof of the theorem 51Y is 41Y21YF1TX71E31Y122M1T and shall be exposed step by step. The action of each operator is shown in detail.

41Y 12H1P121YOFHZ2GZ
21Y 12HN1PZ2GI
21YF
a.
b. 2G
c. 2G
d. 2G12HN1PZ
e. 2G12HN1PZX
f. 2G12HN1PZX
g! 21YOFG121YOFHN1PZX
h! 21YOFG121YOFHN1PZX
21YF1T
a. 21YOFG
b. 21YOFG
c! 21YOFG
41Y21YF1TX
a. 12H1P121YOFHZ2GZ
b. 12H1P121YOFHZ2GZ21YOFG
c. 12H1P121YOFHZ2GZ21YOFGX
7 1272D32H3P31HZM2GZ1GZ
71E
a.
b. 32H3P31HZ
c. 32H3P31HZ1272D
d. 32H3P31HZ1272DM
e. 32H3P31HZ1272DM2GZ1GZ
f! 12H1P13HZ3272DM2GZ3GZ
41Y21YF1TX71E31Y122M
a! 34H3P35HZ5472DM4GZ5GZ
b! 14H1P15HZ5472DM4GZ5GZ
c! 12H1P121YOFHZ21YOF272DM
d! 21YOF272D2GZ21YOFGX
e! 21YOF172D1GZ21YOFGX
41Y21YF1TX71E31Y122M1T
a. 21YOF172D1GZ
b. 21YOF172D1GZ
c 21YOF172D1GZ which is just the statement of the theorem 51Y.

The proof of the theorem 42Y is 42Y75112MN11QC71E9112M. We shall present this proof step by step indicating the action of each proof operator.

42Y 1172D1GZ
7 1272D32H3P31HZM2GZ1GZ
42Y75112M 12H1P12HZ2GZ
42Y75112MN 12HN1P12HA2GA and memorize 12H1P12HZ2GZ
42Y75112MN11Q 12H12HNX1PA2GA

42Y75112MN11QC 12H1P12HZ2GZ
7 1272D32H3P31HZM2GZ1GZ
71E 12H1P13HZ3272DM2GZ3GZ
42Y75112MN11QC71E9112M 1172D1GZ which is the statement of the theorem.

## REFERENCES


[1] Whitehead A. N., Russell B., *Principia mathematica*, Cambridge, 1927.
[2] Bernays P., Fraenkel A., *Axiomatic Set Theory*, North-Holland Pub. Co., Amsterdam, 1958.
[3] Dumitriu A., *Logica polivalentă*, București, 1943.
[4] Lewis C. I., Langford C. H., *Symbolic logic*, New York, 1951.
[5] Barnes D. W., Mack J. M., *An Algebraic Introduction to Mathematical Logic*, Springer-Verlag, New York, 1975.


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