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PIECEWISE CONVEX INTERPOLATION

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1. Let $n \in N$ and the following two systems of $n+1$ real values:

$$0 = x_0 < x_1 < \ldots < x_n = 1 \tag{1}$$

$$0 = y_0,\ y_1,\ \ldots,\ y_n. \tag{2}$$

In the papers [1], [2] it is proved that if $n \geq 1$ and $y_i - y_{i-1} \neq 0,\ i = 1, 2, \ldots, n$ then there exists a polynomial P which assumes at each point x_i the preassigned value y_i and which is piecewise monotone, more precisely:

$$P(x_i) = y_i, \quad i = 0, 1, \ldots, n \tag{3}$$

$$P'(x)(y_i - y_{i-1}) \geq 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n. \tag{4}$$

There are many papers related to the piecewise monotone interpolation; such references can be found in [4], [5].

The purpose of this paper is to prove the existence of a piecewise convex (by order $p = 1$) interpolating polynomial. Our proof uses the Wolibner-Young's theorem [1], [2] concerning the piecewise monotone (convex by order $p = 0$) interpolation, in the same way that the last one uses the Weierstrass approximation theorem.

2. Let $n \geq 1$ and denote by $\Delta_i^2(y)$ the divided difference $[x_i, x_{i+1}, x_{i+2}; y_i, y_{i+1}, y_{i+2}]$ for any $i = -1, 0, 1, \ldots, n-1$ where $x_{-1} < 0,\ x_{n+1} > 1$ are fixed, and $y_{-1} = y_0,\ y_{n+1} = y_n$.

We have the following

THEOREM 1. 1^0. *If* $\triangle_i^2(y) \neq 0,\ i = -1, 0, 1, \ldots, n-2$, *then there exists a polynomial* P *satisfying (3) and*

$$P''(x) \cdot \triangle_{i-2}^2(y) \geq 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n. \tag{5}$$

2^0. *If* $\triangle_i^2(y) \neq 0,\ i = 0, 1, \ldots, n-1$, *then there exists a polynomial* P *satisfying (3) and*

$$P''(x)\, \triangle_{i-1}^2(y) \geq 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n. \tag{6}$$

To prove it we need the following

LEMMA. For any $\varepsilon > 0$ and $\nu \in \{0, 1, \ldots, n-1\}$ there is a polynomial P_{2,x_ν} satisfying

$$\|\sigma_\nu \varphi_{2,x_\nu} - P_{2,x_\nu}\| \leq \varepsilon, \tag{7}$$

and

$$P''_{2,x_\nu}(x) \triangle^2_{i-2}(y) \geq 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n, \tag{8}$$

where

$$\varphi_{p+1,x_\nu}(x) = \begin{cases} 0, & x < x_\nu \\ (x - x_\nu), & x \geq x_\nu \end{cases} \quad (p \in N), \tag{9}$$

$$\sigma_\nu = \text{sign } \Delta^2_{\nu-1}(y),$$

and $\|\cdot\|$ is the uniform norm in $C[0,1]$.

Proof. For ε and ν fixed we choose a sequence:

$$z_0, z_1, \ldots, z_\nu, z'_\nu, z_{\nu+1}, \ldots, z_n \tag{10}$$

with the following properties:

$$|z_i| \leq \varepsilon/3 \ \text{ if } \ i \leq \nu, \ |\sigma_\nu - z_i| \leq \varepsilon/3 \ \text{ if } \ i \geq v+1, \quad \left|\sigma_\nu - z'_\nu\right| \leq \varepsilon/3, \tag{11}$$

$$\begin{aligned} (z_i - z_{i-1}) \triangle^2_{i-2}(y) > 0 \ &\text{ if } \ i \leq \nu \ \text{ or } \ i > \nu + 1, \\ \left(z'_\nu - z_\nu\right) \triangle^2_{\nu-1}(y) &> 0, \\ \left(z_{\nu+1} - z'_\nu\right) \triangle^2_{\nu-1}(y) &> 0. \end{aligned} \tag{12}$$

Applying the Wolibner-Young theorem we get a piecewise monotone polynomial Q_{2,x_ν}, which interpolates the values (10) on the nodes:

$$x_0, \ x_1, \ \ldots, \ x_\nu, \ x'_\nu, \ x_{\nu+1}, \ \ldots, \ x_n \tag{13}$$

where $x_\nu < x'_\nu \leq x_\nu + \varepsilon/(\varepsilon + 3) < x_{\nu+1}$, that is

$$\begin{aligned} Q'_{2,x_\nu}(x)(z_i - z_{i-1}) \geq 0, \ & x \in [x_{i-1}, x_i], \quad i \leq \nu \ \text{ or } \ i > \nu + 1, \\ Q'_{2,x_\nu}(x)\left(z'_\nu - z_\nu\right) \geq 0, \ & x \in \left[x_\nu, x'_\nu\right], \\ Q'_{2,x_\nu}(x)\left(z_{\nu+1} - z'_\nu\right) \geq 0, \ & x \in \left[x'_\nu, x_{\nu+1}\right], \end{aligned} \tag{14}$$

$$Q_{2,x_\nu}(x_i) = z_i, \quad i = 0, 1, \ldots, n, \qquad Q_{2,x_\nu}\left(x'_\nu\right) = z'_\nu. \tag{15}$$

Now $P_{2,x_\nu}(x) = \int_0^x Q_{2,x_\nu}(t)\, dt$ is the desired polynomial.

In order to prove this we make use of (12) and (14) and obtain $Q'_{2,x_\nu}(x) \triangle^2_{i-2}(y) \geq 0$ if $x \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. Thus (8) is verified.

Further, to prove (7) we have:

$$
\begin{aligned}
&\left|\sigma_\nu \varphi_{2,x_\nu}(x) - P_{2,x_\nu}(x)\right| = \\
&= \left|\sigma_\nu \int_0^x \varphi_{1,x_\nu}(t)\ dt - \int_0^x Q_{2,x_\nu}(t)\ dt\right| \\
&\le \int_0^1 \left|\sigma_\nu \varphi_{1,x_\nu}(t) - Q_{2,x_\nu}(t)\right| dt \\
&\le \int_0^{x_\nu} \left|Q_{2,x_\nu}(t)\right| dt + \int_{x_\nu}^{x'_\nu} \left|\sigma_\nu - Q_{2,x_\nu}(t)\right| dt + \int_{x'_\nu}^1 \left|\sigma_\nu - Q_{2,x_\nu}(t)\right| dt.
\end{aligned}
$$

Using (11) and the piecewise monotonicity of Q_{2,x_ν} we obtain

$$
\int_0^{x_\nu} \left|Q_{2,x_\nu}(t)\right| dt \le \varepsilon/3,
$$

$$
\int_{x'_\nu}^1 \left|\sigma_\nu - Q_{2,x_\nu}(t)\right| dt \le \varepsilon/3,
$$

and

$$
\int_{x_\nu}^{x'_\nu} \left|\sigma_c - Q_{2,x_\nu}(t)\, dt\right| \le \left(x'_\nu - x_\nu\right)(1 + \varepsilon/3) \le \varepsilon/3.
$$

These inequalities prove (7). □

Proof of Theorem 1 First we observe that the elementary function of order $p = 1$ on $[0, 1]$

$$
f(x) = \sum_{\nu=0}^{n-1} c_{\nu+1} \cdot \varphi_{2,x_\nu}(x),
$$

where $c_\nu = (x_\nu - x_{\nu-2}) \triangle^2_{\nu-2}(y)$, $v = 1, 2, \ldots, n$ satisfies the conditions of interpolation

$$
f(x_i) = y_i, \quad i = 0, 1, ..., n.
$$

Since $f(x) = \sum_{\nu=0}^{n-1} |c_{\nu+1}| \left(\text{sign } \triangle^2_{\nu-1}(y) \cdot \varphi_{2,x_\nu}(x)\right)$ and $|c_{\nu+1}| > 0$, $v = 0, 1, \ldots, n-1$, it follows that if $\varepsilon > 0$ is small enough there exist $a_{\nu+1} > 0$, $v = 0, 1, \ldots, n-1$ such that the polynomial

$$
P(x) = \sum_{\nu=0}^{n-1} a_{\nu+1} P_{2,x_\nu}
$$

satisfies (3), where P_{2,x_ν} are due to the previous lemma.

Finally the positivity of the coefficients $a_{\nu+1}$ and the piecewise convexity of P_{2,x_ν} assure that (5) holds true.

The proof of the last part of Theorem is similar. $\square$

Remark. Based on the construction of the sequences (10), (13) (for $p = 1$) we obtain a discrete ε-approximation of the function φ_{p,x_ν} such that its associated polynomial P_{p+1,x_ν} ε-approximates φ_{p+1,x_ν} and its derivative of order $p+1$, $P^{(p+1)}_{p+1,x_\nu}$ has the same sign as the divided differences of order p of the values (2) an the nodes (1). This method applied before for $p = 1$ can be adapted to prove (inductively after p) the existence of a piecewise convex of order p $(p > 1)$ interpolating polynomial. $\square$

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