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PIECEWISE CONVEX INTERPOLATION

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1. Let $n \in N$ and the following two systems of $n+1$ real values :

(1) $$0 = x_0 < x_1 < \ldots < x_n = 1$$

(2) $$0 = y_0, y_1, \ldots, y_n.$$

In the papers [1], [2] it is proved that if $n \geqslant 1$ and $y_i - y_{i-1} \neq 0$, $i = 1, 2, \ldots, n$ then there exists a polynomial P which assumes at each point x_i the preassigned value y_i and which is piecewise monotone, more precisely :

(3) $$P(x_i) = y_i, \quad i = 0, 1, \ldots, n$$

(4) $$P'(x)(y_i - y_{i-1}) \geqslant 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n.$$

There are many papers related to the piecewise monotone interpolation ; such references can be found in [4], [5].

The purpose of this paper is to prove the existence of a piecewise convex (by order $p = 1$) interpolating polynomial. Our proof uses the Wolibner-Young's theorem [1], [2] concerning the piecewise monotone (convex by order $p = 0$) interpolation, in the same way that the last one uses the Weierstrass approximation theorem.

2. Let $n \geqslant 1$ and denote by $\Delta_i^2(y)$ the divided difference $[x_i, x_{i+1}, x_{i+2}; y_i, y_{i+1}, y_{i+2}]$ for any $i = -1, 0, 1, \ldots, n-1$ where $x_{-1} < 0$, $x_{n+1} > 1$ are fixed, and $y_{-1} = y_0$, $y_{n+1} = y_n$.

We have the following

THEOREM. 1°. *If* $\Delta_i^2(y) \neq 0$, $i = -1, 0, 1, \ldots, n-2$, *then there exists a polynomial* P *satisfying* (3) *and*

(5) $$P''(x) \cdot \Delta_{i-2}^2(y) \geqslant 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n.$$

2°. *If* $\Delta_i^2(y) \neq 0$, $i = 0, 1, \ldots, n-1$, *then there exists a polynomia* P *satisfying* (3) *and*

(6) $$P''(x)\, \Delta_{i-1}^2(y) \geqslant 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n.$$

To prove it we need the following

LEMMA. *For any* $\varepsilon > 0$ *and* $\nu \in \{0, 1, \ldots, n-1\}$ *there is a polynomial* P_{2,x_ν} *satisfying*

(7) $$\|\sigma_\nu \varphi_{2,x_\nu} - P_{2,x_\nu}\| \leqslant \varepsilon,$$

and

(8) $$P''_{2,x_\nu}(x)\,\Delta^2_{i-2}(y) \geqslant 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, n,$$

where

(9) $$\varphi_{p+1,x_\nu}(x) = \begin{cases} 0, & x < x_\nu \\ (x - x_\nu), & x \geqslant x_\nu \end{cases} \quad (p \in N),$$

$$\sigma_\nu = \operatorname{sign}\ \Delta^2_{\nu-1}(y),$$

and $\|\cdot\|$ *is the uniform norm in* $C[0,1]$.

Proof. For ε anv ν fixed we choose a sequence:

(10) $$z_0, z_1, \ldots, z_\nu, z'_\nu, z_{\nu+1}, \ldots, z_n$$

with the following properties:

(11) $|z_i| \leqslant \varepsilon/3$ if $i \leqslant \nu$, $|\sigma_\nu - z_i| \leqslant \varepsilon/3$ if $i \geqslant \nu+1$, $|\sigma_\nu - z'_\nu| \leqslant \varepsilon/3$,

(12) $$(z_i - z_{i-1})\,\Delta^2_{i-2}(y) > 0 \text{ if } i \leqslant \nu \text{ or } i > \nu + 1,$$
$$(z'_\nu - z_\nu)\Delta^2_{\nu-1}(y) > 0,$$
$$(z_{\nu+1} - z'_\nu)\,\Delta^2_{\nu-1}(y) > 0.$$

Applying the Wolibner-Young theorem we get a piecewise monotone polynomial Q_{2,x_ν}, which interpolates the values (10) on the nodes:

(13) $$x_0, x_1, \ldots, x_\nu, x'_\nu, x_{\nu+1}, \ldots, x_n$$

where $x_\nu < x'_\nu \leqslant x_\nu + \varepsilon/(\varepsilon+3) < x_{\nu+1}$, that is

(14) $$Q'_{2,x_\nu}(x)\,(z_i - z_{i-1}) \geqslant 0, \; x \in [x_{i-1}, x_i], \; i \leqslant \nu \text{ or } i > \nu + 1,$$
$$Q'_{2,x_\nu}(x)(z'_\nu - z_\nu) \geqslant 0, \quad x \in [x_\nu, x'_\nu],$$
$$Q'_{2,x_\nu}(x)(z_{\nu+1} - z'_\nu) \geqslant 0, \quad x \in [x'_\nu, x_{\nu+1}],$$

(15) $$Q_{2,x_\nu}(x_i) = z_i, \; i = 0, 1, \ldots, n, \qquad Q_{2,x_\nu}(x'_\nu) = z'_\nu.$$

Now $P_{2,x_\nu}(x) = \int_0^x Q_{2,x_\nu}(t)\,dt$ is the desired polynomial.

In order to prove this we make use of (12) and (14) and obtain $Q'_{2,x_\nu}(x)\Delta^2_{i-2}(y) \geqslant 0$ if $x \in [x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$. Thus (8) is verified.

Further to prove (7) we have:

$$|\sigma_\nu \varphi_{2,x_\nu}(x) - P_{2,x_\nu}(x)| = \left|\sigma_\nu \int_0^x \varphi_{1,x_\nu}(t)\,dt - \int_0^x Q_{2,x_\nu}(t)\,dt\right| \leqslant$$
$$\leqslant \int_0^1 |\sigma_\nu \varphi_{1,x_\nu}(t) - Q_{2,x_\nu}(t)|\,dt \leqslant \int_0^{x_\nu} |Q_{2,x_\nu}(t)|\,dt +$$
$$+ \int_{x_\nu}^{x'_\nu} |\sigma_\nu - Q_{2,x_\nu}(t)|\,dt + \int_{x'_\nu}^1 |\sigma_\nu - Q_{2,x_\nu}(t)|\,dt.$$

Using (11) and the piecewise monotonicity of Q_{2,x_ν} we obtain

$$\int_0^{x_\nu} |Q_{2,x_\nu}(t)|\,dt \leqslant \varepsilon/3,$$

$$\int_{x'_\nu}^1 |\sigma_\nu - Q_{2,x_\nu}(t)|\,dt \leqslant \varepsilon/3,$$

and

$$\int_{x_\nu}^{x'_\nu} |\sigma_\nu - Q_{2,x_\nu}(t)|\,dt \leqslant (x'_\nu - x_\nu)(1 + \varepsilon/3) \leqslant \varepsilon/3.$$

These inequalities prove (7).

Proof of Theorem. 1°. First we observe that the elementary function of order $p = 1$ on $[0, 1]$

$$f(x) = \sum_{\nu=0}^{n-1} c_{\nu+1} \cdot \varphi_{2,x_\nu}(x),$$

where $c_\nu = (x_\nu - x_{\nu-2})\Delta^2_{\nu-2}(y)$, $\nu = 1, 2, \ldots, n$ satisfies the conditions of interpolation

$$f(x_i) = y_i, \quad i = 0, 1, \ldots, n.$$

Since $f(x) = \sum_{\nu=0}^{n-1} |c_{\nu+1}|\,(\operatorname{sign}\ \Delta^2_{\nu-1}(y) \cdot \varphi_{2,x_\nu}(x))$ and $|c_{\nu+1}| > 0$, $\nu = 0, 1, \ldots, n-1$, it follows that if $\varepsilon > 0$ is small enough there exist $a_{\nu+1} > 0$, $\nu = 0, 1, \ldots, n-1$ such that the polynomial

$$P(x) = \sum_{\nu=0}^{n-1} a_{\nu+1} P_{2,x_\nu}$$

satisfies (3), where P_{2,x_ν} are due to the anterior lemma.

Finally the positivity of the coefficients $a_{\nu+1}$ and the piecewise convexity of P_{2,x_ν} assure that (5) holds true.

The proof of the last part of Theorem is similar.

Remark. Based on the construction of the sequences (10), (13) (for $p = 1$) we obtain a discrete ε-approximation of the function φ_{p,x_ν} such that its associated polynomial P_{p+1,x_ν} ε-approximates φ_{p+1,x_ν} and its derivative of order $p+1$, $P^{(p+1)}_{p+1,x_\nu}$ has the same sign as the divided differences of order p of the values (2) on the nodes (1). This method applied before for $p = 1$ can be adapted to prove (inductively after p) the existènce of a piecewise convex of order p ($p > 1$) interpolating polynomial.

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