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ON SOME KOROVKIN SUBSPACES

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1. Let X be a compact Hausdorff space and $C(X)$ the space of all real-valued continuous functions on X with supremum norm and the natural ordering. For x in X let e_x be the corresponding Dirac functional on $C(X)$. Let H be a closed linear subspace of $C(X)$, linearly separating (i.e. for all $x, y \in X$, $x \neq y$ there exist $f, g \in H$ such that $f(x)g(y) \neq f(y)g(x)$) ; suppose that H is a lattice under the natural ordering.

Let $M_+(X)$ be the set of all positive Radon measures on X and

$$L(H) = \{x \in X : \text{if } \mu \in M_+(X) \text{ and } \mu(h) = h(x) \text{ for all } h \text{ in } H, \text{ then } \mu = e_x\}.$$

We have (see [11]):

(1) $L(H)$ is a closed nonvoid set.

(2) There exists $h \in H$, $h(x) > 0$ for all x in X.

Moreover, for any f in $C(X)$ the restriction $f|L(H)$ has a unique extension in H; let us denote this extension by Tf. Then T is a positive linear operator carrying $C(X)$ onto H, $T^2 = T$ and

(3) $$\{f \in C(X) : Tf = f\} = H.$$

For each $f \in C(X)$ let $Rf = f|L(H)$; then R is a positive linear operator from $C(X)$ onto $C(L(H))$.

Let $V : C(X) \to C(X)$ be a positive linear operator which satisfies one of the following equivalent conditions:

(4) $V \circ T = V$

(5) $\ker T \subseteq \ker V$

(6) $\operatorname{supp}(e_x \circ V) \subseteq L(H)$ for all x in X

(7) $V = A \circ R$, where A is a positive linear operator carrying $C(L(H))$ into $C(X)$.

THEOREM 1. *If $L(H)$ is a G_δ-set, then there exists a function φ in $C(X)$ such that:*

(8) $$T\varphi \leqslant \varphi$$

(9) $$\{x \in X : T\varphi(x) = \varphi(x)\} = L(H).$$

For any φ in $C(X)$ which satisfies (8) and (9) the linear subspace $H[\varphi]$ of $C(X)$, spanned by H and φ, is a Korovkin subspace for V, i.e. if (V_i) is a net of positive linear operators from $C(X)$ into $C(X)$ such that $\lim V_i g = Vg$ for all g in $H[\varphi]$, then $\lim V_i f = Vf$ for all f in $C(X)$.

Proof. Since $L(H)$ is a G_δ-set, the Urysohn Lemma] guarantees the existence of a function φ in $C(X)$ such that $\varphi \geqslant 0$ and

(10) $$\{x \in X : \varphi(x) = 0\} = L(H).$$

Then $T\varphi = 0$, hence $T\varphi \leqslant \varphi$. (9) is a consequence of (10). Now, using Theorem 1.1. from [6], it suffices to show that if $x \in X$ and μ is a positive linear functional on $C(X)$ (that is, a positive Radon measure on X) such that $\mu|H[\varphi] = (e_x \circ V)|H[\varphi]$, then $\mu = e_x \circ V$.

We have $\varphi - T\varphi \in H[\varphi]$, $\varphi - T\varphi \geqslant 0$ and

$$\mu(\varphi - T\varphi) = (e_x \circ V)(\varphi - T\varphi) = e_x(V\varphi - (V \circ T)\varphi) = 0.$$

This shows that $\operatorname{supp} \mu \subseteq \{x \in X : (\varphi - T\varphi)(x) = 0\} = L(H)$. Let f be in $C(X)$. Then Tf and f coincide on $L(H)$, hence on $\operatorname{supp} \mu$; this yields

$$\mu(f) = \mu(Tf) = (e_x \circ V)(Tf) = (e_x \circ V \circ T)(f) = (e_x \circ V)(f).$$

Therefore $\mu = e_x \circ V$, *q.e.d.*

Remark 1. For a generalization of Theorem 1 see [10].

Let now X be a compact metric space and H a closed linear subspace of $C(X)$ which contains the constant functions, separates the points of X and is a lattice under the natural ordering. Then H is linearly separating and $L(H)$ is a G_δ-set; moreover, $L(H)$, the Choquet boundary and the Shilov boundary of H coincide (see [11]), i.e.

(11) $$L(H) = \operatorname{Ch}(H) = \operatorname{Sh}(H).$$

Let T and V be as above.

COROLLARY 1. *There exists φ in $C(X)$ which satisfies (8) and (9). $H[\varphi]$ is a Korovkin subspace for V.*

This result, with V replaced by T, was obtained by F. ALTOMARE in [1].

Example 1. Let K be a metrizable Bauer simplex and $H = A(K)$ the space of all real-valued continuous affine functions on K. Denote by exK the set of extreme points of K. Let V be a positive linear operator from $C(K)$ into $C(K)$ such that

(12) $$\operatorname{supp}(e_x \circ V) \subseteq \operatorname{ex} K \quad \text{for all } x \in K.$$

Let φ be a strictly convex function in $C(K)$. From [1, Es. 1.5] and Corollary 1 above follows that $A(K)[\varphi]$ is a Korovkin subspace for V.

Example 2. Let Ω be an open bounded subset of R^n such that $\Omega = \operatorname{int}(\bar{\Omega})$. Denote by $\partial\Omega$ the euclidean boundary of Ω. Suppose that for any f in $C(\partial\Omega)$ there exists a unique solution of the following Dirichlet problem.

(13) $$\Delta u = 0 \ \text{ in } \ \Omega, \quad u|\partial\Omega = f, \ u \in C(\bar{\Omega}).$$

Let $H = \{u \in C(\bar{\Omega}) : \Delta u = 0 \text{ in } \Omega\}$. Let V be a positive linear operator from $C(\bar{\Omega})$ into $C(\bar{\Omega})$ such that

(14) $$\operatorname{supp}(e_x \circ V) \subseteq \partial\Omega \quad \text{for all } x \in \bar{\Omega}.$$

Let φ be in $C(\bar{\Omega})$, $\varphi \geqslant 0$ such that $\{x \in \bar{\Omega} : \varphi(x) = 0\} = \partial\Omega$. It follows from [1, Es. 1.6] and from Corollary 1 that H is a Korovkin subspace for V.

Example 3. Let $X = [0, 1] \times [0,1]$, $H = \{h \in C(X)$: there exists $(a, b, c, d) \in R^4$ such that $h(x, y) = axy + bx + cy + d$ for all $(x, y) \in X\}$. In this case

$$Tf(x, y) = (1 - x)(1 - y)\ f(0,0) + (1 - x)\ yf\ (0,1) + x(1 - y) f(1,0) + xyf\ (1,1).$$

Let φ_1, φ_2 be two strictly convex functions in $C[0,1]$ and $\varphi(x, y) = \varphi_1(x) + \varphi_2(y)$. Then $H[\varphi]$ is a Korovkin subspace for T (see [1, Es. 1.9).

Let $a_i \in C(X)$, $a_i \geqslant 0$, $i = 1, 2, 3, 4$ and $V : C(X) \to C(X)$, $Vf(x, y) = a_1(x, y)\ f(0, 0) + a_2(x, y)\ f(0, 1) + a_3(x, y)\ f(1,0) + a_4(x, y) f(1, 1)$. By Corollary 1, $H[\varphi]$ is a Korovkin subspace for V.

2. Let now X be a compact metric space, H a linear subspace of $C(X)$ which contains the constant functions and separates the points of X, S the minimum-stable convex cone spanned by H in $C(X)$. In [8] and [9] we have considered the set Sp of all $f \in C(X)$ with the following property :

(15) if $x \in X$, $\mu \in M_+(X)$, $\mu \neq e_x$ and $\mu(s) \leqslant e_x(s)$ for all $s \in S$, then $\mu(f) < e_x(f)$

and the set St of all $f \in C(X)$ for which

(16) if μ, $\nu \in M_+(X)$, $\mu \neq \nu$ and $\mu(s) \leqslant \nu(s)$ for all $s \in S$, then $\mu(f) < \nu(f)$.

From [9, Proposition 5.2] and [4, Corollary 1] it follows

(17) $$\mathrm{Sp} = \mathrm{St} \neq \emptyset.$$

THEOREM 2. *If φ is in St, then $H[\varphi]$ is a Korovkin subspace for any lattice homomorphism P from $C(X)$ into $C(X)$.*

Proof. Let μ be a positive Radon measure on X and $x \in X$ such that $\mu|H[\varphi] = (e_x \circ P)|H[\varphi]$. Let $s \in S$. Then $s = \min(h_1, \ldots, h_n)$ for some $h_1, \ldots, h_n \in H$. Since P is a lattice homomorphism, we obtain $\mu(s) \leqslant (e_x \circ P)(x)$. Now (16) shows that $\mu = e_x \circ P$.

By Theorem 1.1 from [6], $H[\varphi]$ is a Korovkin subspace for P.

Remark 2. By (15) and (17), if φ is in *St* then $\operatorname{Ch}(H[\varphi]) = X$. Theorem 3 from [3] shows that Theorem 2 above holds for any lattice homomorphism $P : C(X) \to F$, where F is an arbitrary real Banach lattice.

Example 4. Let E be a locally convex Hausdorff space and X a compact convex metrizable subset of E. Let $H = A(X)$ be the space of all real-valued continuous affine functions on X. Then (see [8] and [9], (5.39)]):

$$\mathrm{St} = \{f \in C(X) : f \text{ is a strictly concave function}\}$$

Applying Theorem 2 we obtain

COROLLARY 2 ([7, Theorem 3]). *If φ is a strictly convex function in $C(X)$, then $A(X)[\varphi]$ is a Korovkin subspace for the identity operator on $C(X)$.*

This result was proved by H. BAUER, G. LEHA and S. PAPADOPOULOU [2] in the special case when X is a compact convex subset of R^n and by C. A. MICCHELLI [6] in the general case with the additional hypothesis that φ is smooth.

Remark 2. Let X be a compact metric space and H a linearly separating, closed linear subspace of $C(X)$ which is a lattice under the natural ordering; let $1 \in H$. If φ is in St, then φ satisfies (8) and (9) (see [8, Proposition 5]). Hence in this case H is a Korovkin subspace for any lattice homomorphism P and for any positive linear operator V which satisfies one of the equivalent conditions (4)—(7). In particular we have

Example 5. Let K and H be as in Example 1. For $f \in C(X)$ let Tf be the unique extension of $f|\mathrm{ex}K$ in $A(K)$. Let φ be a strictly convex function in $C(K)$. Then $A(K)[\varphi]$ is a Korovkin subspace for T and for the identity operator on $C(K)$. This is a generalization of Theorem 1 from [5].

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