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APPROXIMATION OF TWICE DIFFERENTIABLE FUNCTIONS BY POSITIVE LINEAR OPERATORS

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1. Let X be a compact convex subset of a normed real space. Let $\varphi \in C(X)$ and let H be the linear subspace of $C(X)$ spanned by φ and the continuous affine functions on X.

In this paper the relationship between the convexity properties of φ and the Korovkin properties of H is investigated.

2. Let Y be a compact Hausdorff space, $\psi \in C(Y^2)$, $\psi(x, y) > 0$ for all $x, y \in Y$, $x \neq y$. Let $B(Y)$ be the space of all real-valued bounded functions on Y with supremum norm and let $T_i : C(Y) \to B(Y)$ be a net of positive linear operators such that $\lim \|T_i 1 - 1\| = 0$. Let

(1) $$\mu_i(\psi) = \sup \{ |T_i\psi(\cdot, y)(y)| : y \in Y\} .$$

THEOREM 1 ([5]). *If* $\lim \mu_i(\psi) = 0$, *then*

(2) $$\lim \|T_i f - f\| = 0 \quad \textit{for all } f \in C(Y).$$

3. Let E be a normed real space and let X be a compact convex subset of E. Let $e(x) = \|x\|^2$. Let $T_i : C(X) \to B(X)$ be a net of positive linear operators such that

(3) $$T_i 1 = 1 \text{ and } T_i h = h \text{ for all } i \text{ and all } h \in E^*|X.$$

Example 1. Suppose that E is an inner-product space and let $\psi(x, y) = e(x - y)$. Then $\mu_i(\psi) = \|T_i e - e\|$. Moreover (see [5]):

(4) $$\|T_i f - f\| \leqslant 2\omega (f, \| T_i e - e\|^{1/2}) \text{ for all } f \in C(X)$$

where ω is the modulus of continuity.

Example 2. Let now $E = R^n$ and $p_j(x_1, \ldots, x_n) = x_j$. Let $\varphi \in C(X)$ with the following property:

(5) there exist $a_0, a_1, \ldots, a_{n+1} \in B(X)$ and $k \in R$, $k > 0$ such that

$$\psi(x, y) = a_0(y)\, \varphi (x) + \sum_{j=1}^{n} a_j(y)\, p_j(x) + a_{n+1}(y) \geqslant ke(x - y)$$

for all $x, y \in X$, and

$$a_0\varphi + \sum_{j=1}^{n} a_j p_j + a_{n+1} = 0 \text{ on } X.$$

Then $\mu_i(\psi) = \|(T_i\varphi - \varphi)\, a_0\|$. Moreover (see [5]):

(6) $$\|T_i f - f\| \leqslant 2\omega(f, (k^{-1}\|(T_i\varphi - \varphi)\, a_0\|)^{1/2}) \quad \text{for all } f \in C(X).$$

The case $n = 1$ is investigated in [8] and [3].

4. A function $\varphi \in C(X)$ with the property

(7) There exists $c \in R$, $c > 0$ such that

$$(x, p, y\,;\, \varphi) = (1 - p)\,\varphi(x) + p\,\varphi(y) - \varphi((1-p)x + py) \geqslant cp(1-p)\,e(x-y)$$

for all $x, y \in X$ and all $p \in [0,1]$

is called c-convex.

If $E = R^n$ and φ is a convex function from $C(X)$ having the property (5), then φ is $k/\|a_0\|$-convex.

If $E = R$, $X = [a, b]$, $\varphi \in C[a, b]$ is c-convex and

(8) $$\varphi_r'(a) > -\infty,\ \varphi_l'(b) < +\infty$$

then φ has the property (5) with $a_0 = 1$ and $k = c$.

In particular, for a convex function $\varphi \in C[a, b]$ with (8), the conditions (5) and (7) are equivalent.

5. Let $X \subset U \subset E$, U open and let $f: U \to R$ such that

(9) f is twice differentiable on U and there exists $M > 0$ such that $\|f''(x)\| \leqslant M$ for all $x \in U$.

Let E be an inner-product space. Then the Taylor formula together with (3) imply

(10) $$|(T_i f - f)\ (x)| \leqslant (M/2)\ (T_i e - e)\ (x) \quad \text{for all } x \in X.$$

This yields

(11) $$\|T_i f - f\| \leqslant (M/2)\ \|\ T_i e - e\|.$$

For $E = R$ see [4, § 2.6].

Let now $n = 1$ in Example 2. Let $\varphi \in C\,[a, b]$ be a convex function having the property (5). It is shown in [3] that if $f \in C^2\,[a, b]$, then

(12) $$\|T_i f - f\| \leqslant (2\|a_0\|/k)\ \|T_i\varphi - \varphi\|\,\|f''\|.$$

6. The inequalities (4) and (6) hold for all continuous functions; (10) and (12) hold only for smaller classes of functions, but — for these classes — they are better than (4) and (6). We shall deduce (10) and (12) from a more general result.

Let E be a normed real space and $\varphi \in C(X)$ a c-convex function. Let $x_0 \in X$ and let $\theta_i(f) = T_i f(x_0)$. By (3), θ_i are probability Radon measures having x_0 as barycenter. It is shown in [7] that for all $f \in C(X)$ there exist $x, y \in X$ $x \neq y$ and $p \in (0,1)$ such that

(13) $$\theta_i(f) - f(x_0) = (\theta_i(\varphi) - \varphi(x_0))\ (x, p, y\,;\, f)/(x, p, y\,;\, \varphi).$$

If $M > 0$ let us denote

$$D(M) = \{g \in C(X): |(x, p, y\,;\, g)| \leqslant Mp\,(1-p)e(x-y)$$

for all $x, y \in X$ and all $p \in [0,1]\}$.

From (13) we obtain

(14) $$|(T_i f - f)\ (x_0)| \leqslant (M/c)\ (T_i\varphi - \varphi)\ (x_0) \quad \text{for all } f \in D(M).$$

This yields

(15) $$\|T_i f - f\| \leqslant (M/c)\ \|\ T_i\varphi - \varphi\| \quad \text{for all } f \in D(M).$$

Suppose now that f has the property (9). Then it is easy to verify that $f \in D(M/2)$. Thus we have

THEOREM 2. *If f has the property* (9) *then*

(16) $|(T_i f - f)\ (x)| \leqslant (M/2c)\ (T_i\varphi - \varphi)\ (x)$ *for all* $x \in X$.
In particular,

(17) $$\|T_i f - f\| \leqslant (M/2c)\ \|\ T_i\varphi - \varphi\|.$$

If E is an inner-product space, then e is a 1-convex function, so from (16) and (17) we obtain (10) and (11). Since φ in (12) is $k/\|a_0\|$-convex, from (17) we deduce

(18) $$\|T_i f - f\| \leqslant (\|a_0\|/2k)\|\ T_i\varphi - \varphi\ \|\ \|f''\| \quad \text{for all } f \in C^2[a, b]$$

which is an improved version of (12).

7. Let now X be a compact convex metrizable subset of a locally convex space. Let $\varphi \in C(X)$ and let H be the linear subspace of $C(X)$ spanned by φ and the continuous affine functions on X. The following are equivalent (see [2] and [6]):

(i) H is a Korovkin subspace

(ii) φ is strictly convex or strictly concave.

On the other hand, the c-convexity of φ is a sufficient condition in order to have (14); if E is an inner-product space, then this condition is also necessary.

The signification of the condition $f \in D(M)$ in (14) is illustrated in [4, Corollary 5.2].

8. Let E be an inner-product space. For each $x \in X$ let ν_x be a probability Radon measure on X having x as barycenter. For $n \in N$ and

$1 \leq j \leq n$ let $p_{nj} \in [0,1]$ such that $p_{n1} + \cdots + p_{nn} = 1$. Let $P_n : X^n \to X$, $P_n(x_1, \ldots, x_n) = p_{n1}x_1 + \cdots + p_{nn}x_n$. Let $\mu_{xn} = \nu_x \otimes \cdots \otimes \nu_x$ (n factors). For $f \in C(X)$ and $x \in X$ let

$$B_n f(x) = \int_{X^n} f \circ P_n \, d\,\mu_{xn}$$

$B_n f$ are the Bernstein-Lototski-Schnabl polynomials (see [5]).

Let $F(x) = \nu_x(e)$. It is shown (see [5]) that B_n satisfy (3) and

$$B_n e(x) = \left(\sum_{j=1}^{n} p_{nj}^2\right) F(x) + \left(1 - \sum_{j=1}^{n} p_{nj}^2\right) e(x). \tag{19}$$

Thus we have

$$\|B_n e - e\| = \|F - e\| \sum_{j=1}^{n} p_{nj}^2. \tag{20}$$

From (4) we deduce

$$\|B_n f - f\| \leq 2\,\omega\left(f, \left(\|F - e\| \sum_{j=1}^{n} p_{nj}^2\right)^{1/2}\right) \quad \text{for all } f \in C(X). \tag{21}$$

If $f \in D(M)$, then (14) implies

$$|(B_n f - f)(x)| \leq M(F(x) - e(x)) \sum_{j=1}^{n} p_{nj}^2. \tag{22}$$

In particular, let $E = R$, $X = [0,1]$, $\nu_x(f) = (1-x)f(0) + xf(1)$, $p_{nj} = (1/n)$. Then $B_n f$ are the usual Bernstein polynomials. We have $e(x) = x^2$ and $F(x) = x$. By (13), for each $x \in [0,1]$ and each $f \in C[0,1]$ there exist three distinct points in $[0,1]$ such that

$$B_n f(x) - f(x) = (1/n)x(1-x) \quad [x_1, x_2, x_3\,; f]$$

where $[x_1, x_2, x_3\,; f]$ is the divided difference of f. This is a result from [1].

If $f \in D(M)$, then from (22) we obtain

$$|B_n f(x) - f(x)| \leq (1/n)x(1-x)M$$

for all $x \in [0, 1]$ (see also [5, § 2.7]).

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