MATHEMATICA – REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 14, N° 2, 1985, pp. 137–145

ON INTERPOLATION OPERATORS-IV
(ESTIMATES IN THE NEIGHBOURHOOD OF NODAL POINTS FOR DIFFERENTIABLE FUNCTIONS)

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1. Introduction and results. Many constructive approaches have been made after the celebrated Telyakovskii [8] and Gopengauz's [1] theorems that were established through classical methods. Contributions by Saxena, Freud, Vertesi [2] and, of course, many others in this direction are not beyond comprehension. Recently, in a series of papers [4, 5, 6] we formulated certain identities which formed the actual basis for the construction of interpolatory polynomials leading to the reproduction of Telyakovskii-Gopengauz's theorem [8] for functions whose first derivatives are continuous.

Following an idea put forth by Meir [3], it is possible now to construct a sequence of positive linear interpolary operators which satisfy Telyakovkii-Gopengauz's theorem (and, of course, some other type also in which estimates of the differences can be expressed in terms of the nbds of nodal points).

It is worthwhile to mention that these estimates reflect the fact that the operators are interpolatory :

2. The oparator $Q_{nm}(f;\, x)$ $(m \geqslant 0)$. Let $x = \cos t$, $x_{kn} = \cos t_{kn}$

$$t_{kn} = \frac{k\pi}{n},\ k = \overline{0, n} \tag{2.1}$$

and for $k = \overline{0, 2n-1}$

$$\varphi_{kn}(t) = \frac{\sin(t - t_{kn})\cos\frac{1}{2}(t - t_{kn})}{2n\sin\frac{1}{2}(t - t_{kn})} \tag{2.2}$$

$$= \frac{1}{2n}\left[1 + 2\sum_{j=1}^{n-1}\cos j\,(t - t_{kn}) + \cos n(t - t_{kn})\right].$$

If there is no danger of confusion, we shall write t_k, x_k, $\varphi_k(t)$ etc. in place of t_{kn}, x_{kn}, $\varphi_{kn}(t)$ etc. from now onwards. Denote by

$$\Lambda_N(t) \equiv \Lambda_{(2m+4)n}(t) = \sum_{k=0}^{2n-1} \varphi_k^{2m+4}(t)$$

(2.3) $$= \sum_{\nu=0}^{m+2} \frac{a_{\nu n}}{(2n)^{2m+3}} \cos 2\nu n t$$

where $a_{\nu n}$'s are polynomials in n of degree $< 2m+3$ such that

(2.4) $$(2n)^{2m+3} = \sum_{\nu=0}^{m+2} a_{\nu n}.$$

It can easily be seen that $\Lambda_{(2m+4)n}(t)$ is a polynomial in cost of degree $\leqslant (2m+4)n$ which is precisely non-zero and has its maximum as unity. Therefore its minimum exists for the interval $-1 < x = \cos t \leqslant +1$. i.e., Let it be c_N, i.e.,

(2.5) $$1 > \Lambda_N(t) > c_N.$$

Let us also assume

$$p_{0,m}(x) = \varphi_0^{2m+4}(t)$$

(2.6) $$p_{n,m}(x) = \varphi_n^{2m+4}(t)$$

$$p_{k,m}(x) = \varphi_k^{2m+4}(t) + \varphi_{2n-k}^{2m+4}(t)\,; \quad k = \overline{1,\ n-1}$$

and

(2.7) $$q_{k,m}(x) = \frac{p_{k,m}(x)}{\Lambda_N(t)}\,; \quad k = \overline{0,\ n}.$$

Then for any function f given on $[-1,1]$, we define the operator $Q_{n,m}(f, x)$ as follows

(2.8) $$Q_{nm}(f, x) = \sum_{k=0}^{n} g_{k,m}\, q_{k,m}(x)\,, \quad m \geqslant 0$$

where

(2.9) $$g_{k,m} = \sum_{\nu=0}^{m} \frac{(x-x_k)^\nu}{\nu\,!} f^{(\nu)}(x_k).$$

In the following we prove

THEOREM 1. *$Q_{nm}(f, x)$ is a rational function of order* $\{(2m+4)n+m,\ (2m+4)n\}$ *and*

(i) $Q_{nm}(f, x_k) = f(x_k) \qquad k = \overline{0,\ n}$

(ii) $Q_{nm}^{(\nu)}(f, x_k) = f^{(\nu)}(x_k), \nu = \overline{1,\ m}$

THEOREM 2. *Let* $f^{(\nu)} \in C[-1,1]$ *then for any natural number* n, *we have for* $x \in [-1,1]$

(2.10) $$|Q_{nm}^{(\nu)}(f\,;x) - f^{(\nu)}(x)| \leqslant c_m \left\{\frac{\sqrt{1-x^2}}{n}\right\}^{m-\nu} \omega_f(m)\left(\frac{\sqrt{1-x^2}}{n}\right) \nu = \overline{0,\ m}$$

where c_m is an absolute positive constant and $\omega_f(m)(\cdot)$ is the usual modulus of continuity of $f^{(m)}$.
We observe from (2.6)—(2.4) and (2.2) that the following identity holds:

(2.11) $$Q_{nm}(f, x) \equiv 1, f \equiv 1.$$

It is worth comparing that (2.10) is the stronger version of our previous results established in this series of papers [4, 5, 6] in the explicit sense that we have a constructive and simple proof of Telyakovskii-Gopengauz's theorem. We now do not need the complicated methods to prove our assertions. At a consequence of (2.10) we can state:

THEOREM 3: *Let* $f^{(m)} \in c[-1,1]$ *and* $0<\alpha<1$ *be given, then for any natural number* n, *we have uniformly in* $[-1,1]$

(2.12) $$|Q_{nm}^{(\nu)}(f\,;\ x) - f^{(\nu)}(x)| = 0\left\{\frac{|x-x_r|^\alpha}{n^{1-\alpha}}\right\}^{m-\nu} \omega_{f^{(m)}}\left(\frac{|x-x_r|^\alpha}{n^{1-\alpha}}\right), r=\overline{0,n}$$

for $\nu = \overline{0,\ m}$

We remark that (2.12) explicitly exhibits the interpolatory nature of the operators $Q_{nm}(f\,; x)$. It also shows that the estimates are best possible in the neighbourhood of nodal points.

3. In the following we prove a few assertions given in the form of lemma which we need to establish our theorems.

LEMMA 1: *We have uniformly in* $[-1,1]$

(3.1) $$\sum_{k=0}^{n} |x-x_k|^{m+1-\nu} q_{k,m}(x) = 0\left(\frac{\sqrt{1-x^2}}{n}\right)^{m+1-\nu}, \nu = \overline{0,\ m}$$

and

(3.2) $$\sum_{k=0}^{n} |x-x_k|^{m+1-\nu} |q_{k,m}^{(\nu)}(x)| = 0\left(\frac{\sqrt{1-x^2}}{n}\right)^{m+1-2\nu}.$$

Proof. Owing to the transformations by putting $x = \cos t$, $x_k = \cos t_k$ and using (2.5)—(2.7), we have

$$\sum_{k=0}^{n} |x-x_k|^{m+1-\nu} q_{k,m}(x) = \frac{1}{V_N(t)} \sum_{k=0}^{2n-1} |\cos t - \cos t_k|^{m+1-\nu} \varphi_k^{2m+4}(t)$$

$$= \frac{1}{V_N(t)} \Bigg[\sum_{\substack{k=0\\k\neq j}}^{2n-1} |\cos t - \cos t_k|^{m+1-\nu} \varphi_k^{2m+4}(t)$$

$$+ |\cos t - \cos t_j|^{m+1-\nu} \varphi_j^{2m+4}(t)\Bigg]$$

where j is defined by

$$|t - t_j| \leqslant \frac{\pi}{2n}. \tag{3.3}$$

Making use of

$$|\cos t - \cos t_k| \leqslant 2 \sin t \sin \frac{1}{2} |t| - t_k| + 2 \sin^2 \frac{1}{2} (t - t_k)$$

and

$$(a + b)^p \leqslant 2^{p-1} (a^p + b^p)$$

and denoting

$$v_k = 2n \sin \frac{1}{2} (t - t_k) \tag{3.4}$$

$$|v_k| \geqslant 2i - 1,\ i = |k - j|,\ k \neq j,\ k = \overline{0, 2n-1}$$

$$j = \overline{0, 2n-1}$$

we at once obtain

$$\begin{aligned}
\sum_{k=0}^{n} |x - x_k|^{m+1-\nu} q_{k,m}(x) &\leqslant 2^{m-\nu} \left(\frac{\sin t}{2n}\right)^{m+1-\nu} \sum_{\substack{k=0 \\ k \neq j}}^{2n-1} \frac{1}{|v_k|^{m+3+\nu}} \\
&+ 2^{m-\nu} \left(\frac{1}{2n}\right)^{m+1-\nu} \sum_{\substack{k=0 \\ k \neq j}}^{2n-1} \sin^{2m+2-2\nu} \frac{1}{2} (t - t_k)\, \varphi_k^{2m+4}(t) \\
&+ 2^{m+1-\nu} \left\{ \left(\frac{\sin t}{2n}\right)^{m+1-\nu} + \frac{\sin^{m+1-\nu}(t)}{(4n)^{m+1-\nu}} \right\} \\
&\leqslant \left(\frac{\sin t}{n}\right)^{m+1-\nu} \sum \frac{1}{|v_k|^{m+3+\nu}} + \frac{1}{2^{m+1-\nu}} \left(\frac{\sin t}{n}\right)^{m+1-\nu} \sum \frac{1}{v_k^{2\nu+2}} \\
&+ \left(\frac{\sin t}{n}\right)^{m+1-\nu} \left\{1 + \frac{1}{2^{m+1-\nu}}\right\} \\
&\leqslant \left(\frac{\sin t}{n}\right)^{m+1-\nu} \left\{1 + \frac{1}{2^{m+1-\nu}}\right\} \sum \frac{1}{v_k^{2\nu+2}}.
\end{aligned} \tag{3.5}$$

Here we have made use of

$$|\varphi_k|(t)| \leqslant 1,$$

and

$$|\sin nt| \leqslant |n| \sin t|. \tag{3.6}$$

Thus from (3.5) owing to (3.4), we have (3.1).

In order to establish the second part of the lemma we have, on account of Leibtniz's formula

$$\begin{aligned}
q_{k,m}^{(\nu)}(x) &= \left[\frac{p_{k,m}(x)}{V_N(x)}\right]^{(\nu)} \\
&= \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} p_{k,m}^{(\mu)}(x) \left(\frac{1}{V_N(x)}\right)^{\nu-\mu}.
\end{aligned} \tag{3.7}$$

Now, owing to the repeated use of Markov's inequality for the derivative of the polynomial, we have,

$$|p_{k,m}^{(\mu)}(x)| \leqslant \left(\frac{n}{\sqrt{1-x^2}}\right)^{\mu} |p_{k,m}(x)| \tag{3.8}$$

It is observed that

$$|V_N(x))^{\mu-\nu}| \leqslant e_m \left(\frac{n}{\sqrt{1-x^2}}\right)^{\nu-\mu} \tag{3.9}$$

Thus we have

$$|q_{k,m}^{(\nu)}(x)| \leqslant 2^{\nu} e_m \left(\frac{n}{\sqrt{1-x^2}}\right)^{\nu} p_{k,m}(x) \tag{3.10}$$

On account of (3.10), (3.9), (3.8) and (3.7), we obtain (3.2).

LEMMA 2. *For* $x \in [-1,1]$ *and* $\nu = \overline{0, m}$

$$\sum_{k=0}^{n} |x - x_k|^{m+1-\nu} q_{k,m}(x) = O(|x - x_r|)^{m+1-\nu},\ r = \overline{0, n}\ r \neq k$$

$$\sum_{k=0}^{n} |x - x_k|^{m+1-\nu} q_{k,m}^{(\nu)}(x) = O(|x - x_r|)^{m+1-2\nu},\ r = \overline{0, n}\ r \neq k.$$

The proof depends upon the following inequalities:

$$\sin t \leqslant 2 \sin \left(\frac{t + t_r}{2}\right)$$

and

$$|\sin nt| \leqslant 2n \sin \frac{1}{2} |t - t_r|. \tag{3.11}$$

We make use of Lemma 1 to accomplish the proof.

4. *Proof* the theorem 2 for $\nu = 0$. On account of the identity (2.11), we have from (2.9), (2.8) and (2.7) the identity

$$f(x) - Q_{n,m}(f; x) = \sum_{k=0}^{n} [f(x) - g_{k,m}(x)]\, g_{k,m}(x). \tag{4.1}$$

We make frequent use of

(4.2) $$f(x) - \sum_{\nu=0}^{m} (x-x_k)^{\nu} f^{(\nu)}(x_k) = O\left(|x-x_k|^{m}\, \omega_f(m)(|x-x_k|)\right)$$

which is based on the finite Taylor's expansion. Making the transformation by using $x = \cos t$, $x_k = \cos t_k$ and using (2.9) — (2.6), we obtain from (4.1)

$$f(x) - Q_{n,m}(f;x) = \sum_{\substack{k=0\\ k\neq j}}^{2n-1} (f(\cos t) - g_{k,m}) \frac{\varphi_k^{2m+4}(t)}{\Lambda_N(t)}$$

$$+ \frac{1}{\Lambda_N(t)} (f(\cos t) - g_{j,m})\, \varphi_j^{2m+4}(t) = T_1 + T_2 .$$

In order to estimate the summands T_1 and T_2, we make use of lemma 1, (3.1) for $\nu = 0$, (4.2).

In the similar fashion, we can prove theorem 3 for $\nu = 0$ where we have to use lemma 2 for $\nu = 0$. We distinguish the following two cases separately :

Case I : When $\dfrac{\sqrt{1-x^2}}{n} \leqslant \dfrac{|x-x_r|^{\alpha}}{n^{1-\alpha}}$

and

Case II : When $\dfrac{\sqrt{1-x^2}}{n} > \dfrac{|x-x_r|^{\alpha}}{n^{1-\alpha}}$.

To illustrate the validity of theorem 2 and 3 for arbitrary ν, let us take the particular case of $m = 2$. We have from [6] the identity

$$\Lambda_{8n}(t) = \sum_{k=0}^{n} \varphi_k^8(t) = \frac{1}{7!\,(2n)^7}\Big[\,[309248n^7 - 71680n^5 + 19712\, n^3$$

$$-5280n + \frac{35}{128} + \Big(304886n^7 + 26880n^5 - 22176\, n^3$$

$$+ 7920n - \frac{56}{128}\Big) \cos 2nt + \Big(30720\, n^7 + 43008n^5$$

$$- 3168n + \frac{28}{128}\Big) \cos 4nt + (256n^7 + 1792\, n^5$$

$$+2464\, n^3 + 528\, n - \frac{8}{128}\Big) \cos 6nt + \frac{1}{128} \cos 8nt\Big]$$

$$= \Big[1 - \frac{1}{7!}\Big(\frac{4}{3} \sin^2 nt - \frac{2}{5} \sin^4 nt + \frac{8}{63} \sin^6 nt\Big)\Big] +$$

$$+ \frac{1}{2n^2}\Big[-\frac{2}{3} \sin^2 nt + \frac{2}{3} \sin^4 nt - \frac{8}{90} \sin^6 nt\Big]$$

$$+ \frac{1}{n^4}\Big[\frac{11}{60} \sin^4 nt - \frac{11}{90} \sin^6 nt\Big] -$$

$$- \frac{1}{n^6}\Big[\frac{11}{336} \sin^4 nt + \frac{11}{168} \sin^6 nt\Big] - \frac{\sin^8 nt}{7!\,(2n)^7}.$$

A direct computation gives the inequality

$$1 \geqslant \Lambda_{8n}(t) \geqslant .874.$$

Now, our main aim is to show

(4.4) $$|Q_{n,2}^{(\nu)}(f, x) - f^{(\nu)}(x)| = O\left(\Big(\frac{\sqrt{1-x^2}}{n}\Big)^{2-\nu} \omega_f''\Big(\frac{\sqrt{1-x^2}}{n}\Big)\right) \nu = 1, 2.$$

The version of theorem 3 reads as follows :

(4.5) $$|Q_{n,2}^{(\nu)}(f, x) - f^{(\nu)}(x)| = O\left(\Big(\frac{|x-x_r|^{\alpha}}{n^{1-\alpha}}\Big)^{2-\nu} \omega_f''\Big(\frac{|x-x_r|^{\alpha}}{n^{1-\alpha}}\Big)\right), \nu = 1, 2$$

To prove theorem 1 for $\nu = 1, 2$ we require the following

(4.6) $$\varphi_k'(t_k) = \varphi_k'''(t_k) = \varphi_k^{2p-1}(t_k) = 0 \text{ for, } k = \overline{0,2n-1}$$

and

(4.7) $$\varphi_k''(t_k) \neq 0 = \varphi_k^{2p}(t_k).$$

If we denote by

(4.8) $$b_{k,2}(t) = \frac{\varphi_k^8(t)}{\Lambda_{8n}(t)}$$

then, from (4.6) and (4.7), we have,

(4.9) $$b_{k,2}'(t_k) = 0$$

and

$$b_{k,2}''(t_k) = 8\, \varphi_k''(t_k) - \Lambda_{8n}''(t_k)$$

$$b_{k,2}''(t_j) = 0 ,\ k \neq j.$$

Therefore from (4.9), (4.8) and (4.7), we obtain

(4.10) $$[g_{k,2}(t)\, b_{k,2}(t)]_{t=t_k}' = f'(x_k),\ k = \overline{0,2\, n-1}$$

(4.11) $$[g_{k,2}(t)\, b_{k,2}(t)]_{t=t_j}' = 0,\ k \neq j$$

and

$$[g_{k,2}(t)\, b_{k,2}(t)]_{t=t_k}'' = f(x_k)\, \{ 8\, \varphi_k''(t_k) - \Lambda_{8n}''(t_k)\} + f''(x_k) = f''(x_k).$$

To complete the proof, we shall require

(4.12) $$\sum_{k=0}^{n} |x - x_k|^{3-\nu}\, q_{k,2}(x) = O\left(\frac{\sqrt{1-x^2}}{n}\right)^{3-\nu},\quad \nu = 1,2$$

(4.13) $$\sum_{k=0}^{n} |x - x_k|^{4-\nu}\, |q_{k,2}'(x)| = O\left(\frac{\sqrt{1-x^2}}{n}\right)^{3-\nu},\quad \nu = 1, 2, 3$$

and

$$(4.14)\qquad \sum_{k=0}^{n} |x - x_k|^{4-\nu} |q''_{k,2}(x)| = O\left(\left(\frac{\sqrt{1-x^2}}{n}\right)^{2-\nu}\right), \nu = 1, 2.$$

We sketch proofs of (4.12)–(4.14). From Lemma 1, (3.1), (4.12) can easily be deduced. For (4.13) to establish we once more take (3.3) into consideration and write

$$\sum_{k=0}^{n} |x - x_k|^{4-\nu} q'^{k}_{k,2}(x) = I_1 + I_2$$

where

$$(4.15)\qquad I_1 = \frac{1}{\sin t} \sum_{\substack{k=0\\k\neq j}}^{2n-1} (\cos t - \cos t_k)^{4-\nu} \left[\frac{8\varphi_k^7(t)\,\varphi'_k(t) - \Lambda'_{8n}(t)\,\varphi_k^8(t)}{\Lambda^2_{8n}(t)}\right]$$

and

$$(4.16)\qquad I_2 = \frac{1}{\sin t} (\cos t - \cos t_j)^{4-\nu} \left[\frac{8\varphi_j^7(t)\,\varphi'_j(t) - \Lambda'_{8n}(t)\,\varphi_j^8(t)}{\Lambda_{8n}(t)}\right].$$

For the estimation of (4.15), we have, from (3.4)

$$I_1 \leqslant \frac{2^{7-\nu} n}{\sin t} \sum_{\substack{k=0\\k\neq j}}^{2n-1} \left[\left\{\sin t \sin \frac{1}{2}|t - t_k|\right\}^{4-\nu} + \left\{\sin \frac{1}{2}(t - t_k)\right\}^{8-2\nu}\right] \frac{|\sin^7 nt|}{v_k^8}$$

$$\leqslant \frac{2^{7-\nu} n}{\sin t}\left[\frac{(\sin t)^{4-\nu}}{(2n)^{4-\nu}} \sum_{\substack{k=0\\k\neq j}}^{2n-1} \frac{|\sin^7 nt|}{|v_k|^{4+\nu}} + \frac{1}{(2n)^{8-2\nu}} \sum_{\substack{k=0\\k\neq j}}^{2n-1} \frac{|\sin^7 nt|}{v_k^{2\nu}}\right]$$

$$\leqslant 2^{8-\nu} \frac{n}{\sin t} \left(\frac{\sin t}{2n}\right)^{4-\nu} \sum \frac{1}{v_k^{2\nu}}$$

$$= O\left(\left(\frac{\sqrt{1-x^2}}{n}\right)^{3-\nu}\right), \nu = 1, 2, 3$$

Here we have made use of

$$(4.17)\qquad |\varphi'_k(t)| \leqslant \frac{n}{|v_k|}$$

and

$$|(\Lambda'_{8n}(t)| \leqslant 8n.$$

Similarly, owing to

$$|\varphi''_k(t)| \leqslant \frac{4n^2}{v_k}$$

and

$$|\Lambda''_{8n}(t)| \leqslant 56\, n^2$$

we can easily demonstrate the proof of (4.14).
The same method applies to get the version of theorem 3 for $m = 2$.

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Received 4.III.1984

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