MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 14, N° 2, 1985, pp. 147—151

STARSHAPED SEQUENCES

GH. TOADER
(Cluj-Napoca)

In [4] we have shown that the "hierarchy of convexity" established, for the functions by A.M. BRUCKNER and E. OSTROW in [1], is also valid for sequences. Starting from a property proved in [7] and generalized in [2], we have extended in [6] this hierarchy, inserting between the set of convex sequences and that of starshaped sequences an infinity of sets of sequences. In this paper we prove similar results for starshaped sequences.

Let us recall some definitions and some results from [4] and [6] which we need in what follows.

DEFINITION 1. *A sequence* $(a_n)_{n=0}^{\infty}$ *is called*:

a) convex, if:

(1) $$\Delta^2 a_n = a_{n+2} - 2a_{n+1} + a_n \geqslant 0, \quad \text{for } n \geqslant 0;$$

b) starshaped, if it satisfies:

(2) $$D^1 a_n = \frac{a_{n+1} - a_0}{n+1} - \frac{a_n - a_0}{n} \geqslant 0, \quad \text{for } n \geqslant 1;$$

c) superadditive, if:

(3) $$a_{n+m} + a_0 \geqslant a_n + a_m, \quad \text{for any } n \text{ and } m.$$

DEFINITION 2. *The sequence* $(a_n)_{n=0}^{\infty}$ *has the property "P" in the mean, if the sequence* $(A_n)_{n=0}^{\infty}$ *has the property "P", where*:

(4) $$A_n = \frac{a_0 + \ldots + a_n}{n+1}, \quad \text{for } n \geqslant 0.$$

REMARK 1. Denoting by S_1, S_2, S_3, S_4, S_5 and S_6 the sets of convex, mean-convex, starshaped, superadditive, mean-starshaped, respectively mean-superadditive sequences, we have proved in [4] the proper inclusions:

(5) $$S_1 \subset S_2 \subset S_3 \subset S_4 \subset S_5 \subset S_6.$$

As it is shown in [3], N. OZEKI has proved that $S_1 \subset S_2$ and, if $a_0 = 0$, $S_1 \subset S_3$. As a matter of fact, we attached a_0 in (2) and (3) just to allow $a_0 \neq 0$ in (5).

REMARK 2. Instead of the arithmetic mean (4), in [7] is considered the weighted mean:

$$\bar{A}_n = \frac{p_0 a_0 + \ldots + p_n a_n}{p_0 + \ldots + p_a} \tag{6}$$

with $p_n > 0$ for $n \geqslant 0$. The result from [7] was put in [6] in the following simpler form: the sequence $(\bar{A}_n)_{n=0}^{\infty}$ is convex for any convex $(a_n)_{n=0}^{\infty}$ if and only if there is an $u > 0$ such that:

$$p_n = p_0 \binom{u + n - 1}{n}, \text{ for } n \geqslant 1 \tag{7}$$

where:

$$\binom{v}{0} = 1, \binom{v}{n} = \frac{1}{n!} \prod_{k=0}^{n-1} (v - k), \text{ for } n \geqslant 1, v \in \mathbf{R}. \tag{8}$$

In this case:

$$\bar{A}_n = A_n^u = \frac{\sum_{k=0}^{n} \binom{u+k-1}{k} a_k}{\binom{u+n}{n}} \tag{9}$$

DEFINITION 3. *The sequence $(a_n)_{n=0}^{\infty}$ is u-mean-convex if $(A_n^u)_{n=0}^{\infty}$ is convex. The set of all u-mean-convex sequences is denoted by S_2^u.*

In [6] we have proved:

THEOREM 1. *If $0 < v < u$, then hold the strict inclusions:*

$$S_1 \subset S_2^u \subset S_2^v \subset S_3. \tag{10}$$

For some fixed functions $f,\ g,\ h,\ k : \mathbf{N} \to \mathbf{R}$, let us denote by:

$$T a_n = f(n)\, a_{n+2} + g(n)\, a_{n+1} + h(n) \cdot a_n + k(n) \cdot a_0 \tag{11}$$

and consider the set:

$$S = \{(a_n)_{n=0}^{\infty} : T a_n \geqslant 0, \forall\, n \geqslant 0\}. \tag{12}$$

We have proved in [5] the following general result:

LEMMA 1. *If $(c \cdot n)_{n=0}^{\infty} \in S$ for any real c and if (6) gives an $(\bar{A}_n)_{n=0}^{\infty}$ in S for any $(a_n)_{n=0}^{\infty}$ from S, then there is an $u > 0$ such that the weights p_n be given by (7).*

CONSEQUENCE. *If $(\bar{A}_n)_{n=0}^{\infty}$ given by (6) is starshaped for any starshaped $(a_n)_{n=0}^{\infty}$, then the weights p_n are given by (7) with $u > 0$ adequately chosen.*

DEFINITION 4. *The sequence $(a_n)_{0=n}^{\infty}$ is u-mean-starshaped if $(A_n^u)_{n=0}^{\infty}$, given by (9) is starshaped. The set of u-mean-starshaped sequences is denoted by S_5^u.*

LEMMA 2. *The sequence $(a_n)_{n=0}^{\infty}$ is u-mean-starshaped if and only if it may be represented by:*

$$a_0 = c_0,\ a_1 = (1 + 1/u) \cdot c_1 - c_0/u,\ a_n = \left(1 + \frac{n}{u}\right) \cdot c_n + n\left(1 + \frac{1}{u}\right) \cdot \sum_{k=1}^{n-1} \frac{c_k}{k} - \left(n - 1 + \frac{n}{u}\right) \cdot c_0,\ \text{for } n \geqslant 2 \tag{13}$$

where $c_k \geqslant 0$ for $k \geqslant 2$.

Proof. As it is proved in [4], a sequence $(a_n)_{n=0}^{\infty}$ is starshaped if and only if it may be represented by:

$$a_n = n \sum_{k=1}^{n} \frac{b_k}{k} - (n-1) \cdot b_0 \tag{14}$$

with $b_k \geqslant 0$ for $n \geqslant 2$. So, $(a_n)_{n=0}^{\infty}$ is u-mean-starshaped if and only if $(A_n^u)_{n=0}^{\infty}$ may be represented by:

$$A_n^u = n \sum_{k=1}^{n} \frac{c_k}{k} - (n-1) \cdot c_0 \tag{14'}$$

with $c_k \geqslant 0$ for $k \geqslant 2$. But, from (9), we have:

$$\binom{u+n-1}{n} a_n = \binom{u+n}{n} A_n^u - \binom{u+n-1}{n-1} \cdot A_{n-1}^u$$

that is:

$$a_n = \left(1 + \frac{n}{u}\right) \cdot A_n^u - \frac{n}{u} \cdot A_{n-1}^u. \tag{15}$$

From (14′) and (15) we get (13).

REMARK 3. It was proved in [4] that if the sequence $(a_n)_{n=1}^{\infty}$ is represented by (14), then:

$$\Delta^2 a_n = b_{n+2} - \frac{n}{n+1} b_{n+1}. \tag{16}$$

It is easy to check also the following results:

LEMMA 3. *If the sequence $(a_n)_{n=0}^{\infty}$ is represented by (13), then:*

$$\Delta^2 a_0 = \left(1 + \frac{2}{u}\right) \cdot c_2;\ \Delta^2 a_n = \left(1 + \frac{n+2}{u}\right) \cdot c_{n+2} - \frac{n}{n+1}\left(1 + \frac{2n+3}{u}\right) c_{n+1} + \frac{n-1}{u} c_n,\ \text{for } n > 0. \tag{17}$$

THEOREM 2. *If $0 < v < u$, then the strict inclusions hold:*

$$S_3 \subset S_5^u \subset S_5^v. \tag{18}$$

Proof. (*i*) Let us suppose that the sequence $(a_n)_{n=0}^{\infty}$ is represented as in (13) and also as in (14). This may be done for every sequence. Then, from (16) and (17) we deduce:

$$b_2 = \left(1 + \frac{2}{u}\right)c_2$$

and

$$b_{n+2} - \frac{n}{n+1}\, b_{n+1} = \left(1 + \frac{n+2}{u}\right)c_{n+2} - \frac{n}{n+1}\left(1 + \frac{2n+3}{u}\right)c_{n+1} + \frac{n-1}{u}\, c_n$$

for $n \geq 1$, which give, by induction:

$$c_n = \frac{u}{u+n}\, b_n + \frac{n(n-2)}{(u+n)\,(n-1)}\, c_{n-1}, \quad \text{for } n \geq 2.$$

So, if $b_n \geq 0$ for $n \geq 2$, we obtain, step by step, $c_n \geq 0$ for $n \geq 2$. That is $S_3 \subset S_5^u$. The inclusion is proper because we have:

$$b_3 = \left(1 + \frac{3}{u}\right)c_3 - \frac{2}{3u}\, c_2$$

which yields $b_3 < 0$ for $c_2 = 1$ and $c_3 = 0$.

(*ii*) Now suppose that the sequence $(a_n)_{n=0}^{\infty}$ is represented by (13) and by:

(13′) $a_0 = d_0;\ a_1 = (1 + 1/v)\, d_1 - d_0/v;\ a_n = (1 + n/v)\, d_n +$

$$+\, n\left(1 + \frac{1}{v}\right)\sum_{k=1}^{n-1}\frac{d_k}{k} - \left(n - 1 + \frac{n}{v}\right) d_0, \quad \text{for } n \geq 2.$$

From (17) we have:

$$(1 + 2/u)\cdot c_2 = (1 + 2/v)\cdot d_2$$

and for $n \geq 1$:

$$\left(1 + \frac{n+2}{u}\right)\cdot c_{n+2} - \frac{n}{n+1}\left(1 + \frac{2n+3}{u}\right)c_{n+1} + \frac{n-1}{u}\cdot c_n =$$

$$= \left(1 + \frac{n+2}{v}\right) d_{n+2} - \frac{n}{n+1}\cdot\left(1 + \frac{2n+3}{v}\right)\cdot d_{n+1} + \frac{n-1}{v}\, d_n.$$

So, again by induction:

$$d_n = \frac{v(u+n)}{u(v+n)}\cdot c_n + \frac{v(u-v)}{u}\,\frac{n!}{n-1}\sum_{k=2}^{n-1}\frac{(k-1)\,c_k}{k!\cdot(v+k)\,\ldots\,(v+n)}.$$

Since $u > v$, if $c_k \geq 0$ for $n \geq 2$, then $d_n \geq 0$ for $n \geq 2$. Hence, by Lemma 2, $S_5^u \subset S_5^v$. The inclusion is proper because:

$$c_3 = \frac{u(v+3)}{v(u+3)}\, d_3 + \frac{3}{2}\,\frac{u(v-u)}{v(u+2)\,(u+3)}\, d_2$$

and $d_3 = 0$, $d_2 > 0$ give $c_3 < 0$.

REMARK 4. For $u = 1$, $S_5^u = S_5$, so that $S_4 \subset S_5^1$. By Theorem 2, we have also:

(19) $$S_4 \subset S_5^u, \text{ for } 0 < u < 1.$$

As we shall prove by the following examples, there is no inclusion between S_4 and S_5^u for $u > 1$.

Example 1. For $a_n = [n/2]$, where $[x]$ denotes the integer part of x, we have:

$$D^1 A_2^u = \frac{u(1-u)}{6(u+2)\,(u+3)}$$

so that $(a_n)_{n=0}^{\infty}$ is in S_4 (see [4]) but not in S_5^u for $u > 1$.

Example 2. For an arbitrary sequence of the form (13) we have:

$$a_n + a_0 - a_{n-1} - a_1 = \left(1 + \frac{n}{u}\right)\cdot c_n + \frac{u - n^2 + 3n - 1}{u(n-1)}\, c_{n-1} +$$

$$+ \left(1 + \frac{1}{u}\right)\sum_{k=2}^{n-2}\frac{c_k}{k}.$$

For any $u > 0$ there is an n_0 such that $n_0^2 - 3n + 1 > u$. Hence, if we take $c_k = 0$ for $k \neq n_0 - 1$ and $c_{n_0-1} = 1$, we get a sequence $(a_n)_{n=0}^{\infty}$ in S_5^u but not in S_4.

REFERENCES

[1] Bruckner, A. M., Ostrow, E., *Some function classes related to the class of convex functions*, Pacific J. Math., **12**, 1203–1215 (1962).

[2] Lacković, I. B., Simić, S. K., *On weighted arithmetic means which are invariant with respect to k-th order convexity*, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz., 461–497, 159–166 (1974).

[3] Mitrinović, D. S., Lacković, I. B., Stanković, M. S., *On some convex sequences connected with N. Ozeki's results*, Univ. Beograd. Publ. Elektrotehn. Fak., Ser. Mat. Fiz. 634–677, 3–24 (1979).

[4] Toader, Gh., *A hierarchy of convexity for sequences*. L'analyse Num. et la Th. de l'Approx. **12**, 2, 187–192 (1983).

[5] Toader, Gh., *A transformation that preserves some sequence classes*. Sem. "Th. Angheluţă", Lucr. Ses. Comun. St., Cluj-Napoca, 219–222 (1983).

[6] Toader, Gh., *On some properties of convex sequences*. Submitted for publication.

[7] Vasić, P.M., Kečkić, J. D., Lacković, I. B., Mitrović, Ž. M., *Some properties of arithmetic means of real sequences*, Mat. Vesnik, **9** (**24**), 205–212 (1972).

Received 18.XII.1984

Institutul Politehnic
Catedra de Matematici
3400 Cluj-Napoca
ROMANIA