MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 14, N° 2, 1985, pp. 153 — 158

THE STOCHASTIC BOTTLENECK TRANSPORTATION PROBLEM

ŞTEFAN ŢIGAN and I. M. STANCU-MINASIAN
(Cluj-Napoca) (Bucharest)

Abstract. In this paper the minimum risk approach for the bottleneck transportation problem is formulated. It is shown that this problem can be reduced, under certain hypotheses, to a deterministic bottleneck transportation problem.

1. Introduction. We consider here the bottleneck transportation problem

$$\text{minimize } t^{\circ}(X) = \max \{t_{ij}/(i,j) \in L(X)\}$$

subject to

(1) $$\sum_{j=1}^{n} x_{ij} = a_i,\ i \in M = \{1, 2, \ldots, m\}$$

(2) $$\sum_{i=1}^{m} x_{ij} = b_j,\ j \in Q = \{1, 2, \ldots, n\}$$

(3) $$x_{ij} \geqslant 0,\quad (i, j) \in M \times Q,$$

(4) $$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j,$$

where

a_i is the available amount at the i^{th} supply point,
b_j is the requirement at the j^{th} demand point,
t_{ij} is the transportation time from supply point i to demand point j,
x_{ij} is the amount to be transportated from the i^{th} supply point to the j^{th} demand point,
$X = (x_{ij})$ is an element of the set of all feasible solutions of the classical transportation problem

$L(X) = \{(i, j) \in M \times Q / x_{ij} > 0\}$ is the positive graph of X,

and all of the data are integers (or equivalently rationals). The transportation time is independent of the amount of commodity shipped from suppliers to consumers.

The time minimizing (bottleneck) transportation problem was first considered by Barsov (1959) and its study was subsequently developed by Hammer (1969, 1971), Garfinkel and Rao (1971), Szwarc (1965, 1966, 1970, 1971), Bhatia, Swarup and Puri (1977), Derigs and Zimmermann (1978), Zimmermann (1978, 1980), Achary and Seshan (1981), Grabowski (1964a, 1964b), Janicki (1969), Niestierow (1962), Sharma and Swarup (1977, 1978), Sharma (1978), Ramakrishnan (1977), Srinivasan and Thompson (1976).

The solution method proposed by Barsov (1959) was based on the simplex method. Primal methods equivalent to Barsov's method were considered by Szwarc (1971), Hammer (1969, 1971) and Srinivasan and Thompson (1976). A second type method was proposed by Garfinkel and Rao (1971). Their approach is based on the Hungarian method and it involves the introducing of a sufficiently large cost on certain routes. Niestierow (1962) used an adaptation of Kantorowitch's linear programming dual method for solving problem (1) — (4). Also, in [13] is given a method by I. W. Romanowski based on the reduction of problem (1)—(4) to a classical transportation problem. Grabowski (1964 a, 1964 b) transforms the problem into a single classical transportation problem too. Zimmermann (1980) established that both linear and bottleneck objectives can be subsumed under a linear objective over a certain algebraic structure. In [28] the primal method for solving algebraic transportation problems which is based on duality principles over semigroups is applied to the bottleneck transportation problem. Achary and Seshan (1981) considered a time minimizing transportation problem with quantity dependent time. In [14] it is shown that the special nature of the objective function enables one to construct very easily an optimal basic feasible solution in finite number of steps using only the maximum flow routine of the well-known primal-dual algorithm of Ford and Fulkerson. Sharma and Swarup (1978) developed finite procedures for two- and three-dimensional time minimizing transportation problem. These procedures are based on moving from one basic solution to another until optimality is reached. The three-dimensional time minimizing transportation problem has been also studied by Sharma and Swarup (1977), Sharma (1978), Maggu and Sharma (1980).

2. The minimum risk approach for the bottleneck transportation problem.

Let now t_{ij} assume random values:

(5) $$t_{ij} = t'_{ij} + t(\omega)\, t''_{ij},\ \forall\, (i, j) \in M \times Q$$

where t'_{ij}, t''_{ij} $(i \in M,\ j \in Q)$ are constants and $t(\omega)$ is a random variable on a probability space (Ω, K, P), with the continuous and strictly increasing distribution function $T(z)$. Similarly with Bereanu (1963) (see, also [19] and [20]) we consider the minimum risk problem corresponding to level z, associated to the bottleneck transportation problem (1) — (4). This problem consists in finding the optimal solution of the following problem:

Find

(6) $$v(z) = \max P\{\omega / \max_{(i,j) \in L(X)} (t'_{ij} + t(\omega)\, t''_{ij}) \leqslant z\}$$

subject to constraints (1) — (4).

We assume that

(7) $t''_{ij} \neq 0$ for all $(i, j) \in M \times Q$, and all have the same sign. We will show that, under the assumption (7), the minimum risk problem (6) can be solved by a deterministic bottleneck transportation problem which does not depend on the distribution function of the random variable $t(\omega)$.

THEOREM 1: *If the assumption (7) holds and if the distribution function T of $t(\omega)$ is continuous and strictly increasing, then the minimum risk solution of problem (6) does not depend on T and it can be obtained by solving one of the bottleneck transportation problems:*

$$\max \min_{(i,j) \in L(X)} \frac{z - t'_{ij}}{t''_{ij}},\ \text{if } t''_{ij} > 0$$

subject to constraints (1) — (4), or

$$\min \max_{(i,j) \in L(X)} \frac{z - t'_{ij}}{t''_{ij}},\ \text{if } t''_{ij} < 0,$$

subject to constraints (1) — (4).

Proof: From (5) we get:

$$F(X, z) = P\{\omega / \max_{(i,j) \in L(X)} (t'_{ij} + t(\omega)\, t''_{ij}) \leqslant z\} =$$

$$= P\{\omega / (t'_{ij} + t(\omega)\, t''_{ij}) \leqslant z,\ \forall\, (i, j) \in L(X)\}.$$

Hence, according to (7), it results:

$$F(X, z) = \begin{cases} P\{\omega / t(\omega) \leqslant g_{ij},\ \forall\, (i, j) \in L(X)\} & \text{if } t''_{ij} > 0 \\ P\{\omega / t(\omega) \geqslant g_{ij},\ \forall\, (i, j) \in L(X) & \text{if } t''_{ij} < 0 \end{cases}$$

where $g_{ij} = (z - t'_{ij})/t''_{ij}$.

Also, we have:

$$F(x, z) = \begin{cases} P\{\omega / t(\omega) \leqslant \min_{(i,j) \in L(X)} g_{ij}\}, & \text{if } t''_{ij} > 0 \\ P\{\omega / t(\omega) \geqslant \max_{(i,j) \in L(X)} g_{ij}\}, & \text{if } t''_{ij} < 0 \end{cases} =$$

$$= \begin{cases} T(\min_{(i,j) \in L(X)} g_{ij}), & \text{if } t''_{ij} > 0 \\ 1 - T(\max_{(i,j) \in L(X)} g_{ij}), & \text{if } t''_{ij} < 0. \end{cases}$$

Then the problem (6) becomes

$$\max F(X, z) = \begin{cases} \max T(\min_{(i,j) \in L(X)} g_{ij}), & \text{if } t''_{ij} < 0 \\ 1 - \min T(\max_{(i,j) \in L(X)} g_{ij}), & \text{if } t''_{ij} < 0 \end{cases}$$

hence, by the assumption that T is continuous and strictly increasing, we get:

$$v(z) = \max F(x,z) = \begin{cases} T(\max \min\limits_{(i,j)\in L(X)} g_{ij}), & \text{if } t''_{ij} > 0 \\ 1 - T(\min \max\limits_{(i,j)\in L(X)} g_{ij}), & \text{if } t''_{ij} < 0. \end{cases}$$

Thus, one gets immediately the theorem.

Now we assume that

(8) $$t_{ij} = t'_{ij} + t^*_{ij}(\omega)\, t''_{ij}, \forall (i,j) \in N \times Q,$$

where $t^*_{ij}((i,j) \in M \times Q)$ are independent random variables with the continuous and strictly increasing distribution functions T_{ij}.

Also, we assume that:

(9) $$t''_{ij} > 0, \forall (i,j) \in M \times Q.$$

In this case, the minimum risk solution of problem (6) depends on T_{ij}.

Indeed, as in the previous case, we have:

$$F(x,z) = P\{\omega / \max_{(i,j)\in L(X)} (t'_{ij} + t^*_{ij}(\omega)\, t''_{ij}) \leqslant z\} =$$

$$= P\{\omega / (t'_{ij} + t^*_{ij}(\omega)\, t''_{ij}) \leqslant z, \forall (i,j) \in L(x)\} =$$

$$= P\{\omega / t^*_{ij}(\omega) \leqslant g_{ij}, \forall (i,j) \in L(x)\} =$$

$$= \prod_{(i,j)\in L(X)} T_{ij}(g_{ij}), \text{ where } g_{ij} = (z - t'_{ij})/t''_{ij}.$$

Hence

(10) $$\max F(X,z) = \max \prod_{(i,j)\in L(X)} T_{ij}(g_{ij})$$

subject to constraints (1) – (4).

This problem is equivalent with the problem:

(11) $$\max \ln \Big(\prod_{(i,j)\in L(X)} T_{ij}(g_{ij}) \Big),$$

subject to constraints (1) – (4), or

(12) $$\max \sum_{(i,j)\in L(X)} \ln T_{ij}(g_{ij})$$

subject to constraints (1) – (4).

Hence we have:

THEOREM 2: *If the assumption (9) holds and if the distribution functions T_{ij} of $t^*_{ij}(\omega)$ are continuous and strictly increasing, then the minimum risk solution of the problem*

$$\max P\{\omega / \max_{(i,j)\in L(X)} (t'_{ij} + t^*_{ij}(\omega)\, t''_{ij}) \leqslant z\}$$

subject to constraints (1) – (4), can be obtained by solving the problem (10) or, equivalently, the problem (12).

The problem (12) is a fixed charge transportation problem, which however depends on the distribution functions T_{ij} of the random variables $t^*_{ij}(\omega)$.

3. Remarks. A similar approach can be applied to the classical botleneck assignment problem or to the bottleneck assignment problem with an additional constraint (see [25]).

We mention also that another type of stochastic transportation and bottleneck assignment problems was studied by Yechiali (1968, 1971).

REFERENCES

[1] K. K. Achary, C. R. Seshan (1981): *A Time Minimizing Transportation Problem with Quantity Dependent Time*, European J. Oper. Res., **7** (3).

[2] A.R. BARSOV (1959): *What is Linear Programming*, Moscow 90–101 (in Russian).

[3] B. Bereanu (1963): *On stochastic linear programming, I: Distribution problems. A single random variable*, Rev. Roumaine Math. Pures Appl., **8**, 683–697.

[4] H. L. Bhatia, K. Swarup, M. C. Puri (1977): *A Procedure for Time Minimization Transportation Problem*, Indian J. Pure Appl. Math., **8** (8), 920–929.

[5] U. Derigs, U. Zimmermann (1978): *An Augmenting Path Method for Solving Linear Bottleneck Transportation Problem*, Computing, **19**, 285–295.

[6] R. S. Garfinkel, M. R. Rao (1971): *The Bottleneck Transportation Problem*, Naval Res. Logist. Quart., **18** (4), 465–472.

[7] W. Grabowski (1964 a): *Problem of Transporatation in Minimum Time*, Bull. Acad. Polon. Sci., **12**, 107–108.

[8] W. Grabovschi (1964b) *Transportation Problem with Minimization of Time*, Przeglad Statyst., **11**, 333–359.

[9] P. L. Hammer (1969): *Time Minimizing Transportation Problem*, Naval Res. Logist. Quart., **16**, 345–357.

[10] P. L. Hammer (1971): *Communication on "The Bottleneck Transportation Problem" and "Some Remarks on the Time Transportation Problem"*, Haval Res. Logist. Quart., **18** (4), 487–490.

[11] A. Janicki (1969): *The Time Transportation Problem*, M. S. Thesis, University of Wroclaw, Poland (in Polish).

[12] P. L. Maggu, J. K. Sharma (1980): *A Procedure for Time Minimizing Solid Transportation Problem*, Pure Appl. Math. Sci., **12** (1–2), 19–27.

[13] E. P. Niestierow (1962): *Transportation Problems in Linear Programming*, (Moskow) 72–80 (in Russian).

[14] C. S. Ramakrishnan (1977): *A Note on the Time Minimizing Transportation Problem*, Opsearch, **14** (3), 207–209.

[15] J. K. Sharma, K. Swarup (1978): *Time Minimization in Transportation Problems*, New Zealand Oper. Res., **6** (1), 75–88.

[16] J. K. Sharma, K. Swarup (1977): *The Time Minimizing Multidimensional Transportation Problem*, Journal of Eng. Production, **1**, 121–129.

[17] J. K. Sharma (1978): *A Note on the Time Minimizing Solid Transportation Problem*, Pure Appl. Math. Sci., **7** (1–2), 41–42.

[18] V. Srinivasan, G. L. Thompson (1976): *Algorithms for minimizing total cost, bottleneck time and bottleneck shipment in transportation problems*, Naval Res. Logist. Quart. **23**, (4), 567–595.

[19] I. M. Stancu-Minasian (1980): *Programarea stocastică cu mai multe funcţii obiectiv*, Ed. Academiei, Bucureşti.

[20] I. M. Stancu-Minasian, S. Tigan, *The minimum risk approach to special problems of mathematical programming. The distribution function of the optimal value*, Mathematica – Rev. Analyse Num. Th. Approx., **13**, 2, 1984, 175–187.

[21] W. Szwarc (1971): *Some Remarks on the Time Transportation problem*, Naval Res. Logist. Quart., **18** (4), 473–485.

[22] W. Szwarc (1966): *The Time Transportation Problem*, Zastos. Mat., **8**, 231–242.

[23] W. Szwarc (1965): *Das Transportzeit problem*, Mathematik und Kybernetic in der Oekonomie, Academie Verlag Berlin, **19**, 72–78.

[24] W. Szwarc (1970): *The Time Transportation Problem*, Carnegie-Mellon University, Pittsburgh, Pennsylvania, Tehnical Report No. 201.
[25] S. Ţigan (1969): *Sur une problème d'affectation*, Mathematica, **11 (34)**, 1, 163–166.
[26] U. Yechiali (1968): *A Stochastic Bottleneck Assignment Problem*, Management Sci, **14** (11), 732–734.
[27] U. Yechiali (1971): *A Note on a Stochastic Production-Maximizing Transportation Problem*, Naval Res. Logist. Quart., **18** (3,) 429–431.
[28] U. Zimmermann (1980): *Duality for Algebraic Linear Programming*, Linear Algebra Appl., **32**, 9–31.
[29] U. Zimmermann (1978): *A Primal method for Solving Algebraic Transportation Problems Applied to the Bottleneck Transportation Problem*, Proc. of the Polish-Danish Mathematical Programming Seminar. Part one. Edited by J. Krarup and S. Walukiewicz. Warszawa, 139–153.

Received 18.XII.1984

Centrul Teritorial de Calcul,
Str. Republicii, nr. 107,
3400 Cluj-Napoca
ROMANIA
Laboratoarele Catedrei de Cibernetică Economică,
Calea Dorobanţi, nr. 15–17,
71131 Bucureşti
ROMANIA