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APPROXIMATION OF OPTIMAL CONTROL PROBLEMS GOVERNED BY NONLINEAR EQUATIONS

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1. Introduction. In [1] the following optimal control problem was considered : find a pair (y, u) that minimizes the functional

$$\int_0^T (\varphi(y(t)) + g(u(t)))dt + \psi(y(T)) \text{ subject to} \tag{1.1}$$

the state equation

$$\begin{cases} y' - \Delta y + \beta(y) \ni uBy & \text{a.e. in } Q = \Omega \times]0, T[\\ y(x, 0) = y_0(x) & \text{a.e. } x \in \Omega \\ y(x, t) = 0 & \text{for } (x, t) \in \Sigma = \partial\Omega \times]0, T[\end{cases} \tag{1.2}$$

where Ω is an open, bounded domain of R^N with smooth boundary $\partial\Omega$; $\varphi, \psi : L^2(\Omega) \to R$ are nonnegative, locally Lipschitz functions, and g is a proper, lower semicontinuous, convex function that satisfies the growth condition : $\exists C_1 > 0$, $C_2 \in R$ such that

$$g(x) \geqslant C_1 x^2 + C_2 \text{ for all } x \in R \tag{1.3}$$

In (1.2) y' means the derivative of y as a function of t from $[0, T]$ to $L^2(\Omega)$, Δ is the Laplace operator in $L^2(\Omega)$, $\beta \subset R \times R$ is a maximal monotone operator such that $0 \in \beta(0)$, $B : L^2(\Omega) \to L^2(\Omega)$ is a bounded linear operator and u is a scalar function from $L^2(0, T)$.

As we have seen in [1] equation (1.2) can be written as

$$\begin{cases} y' + \partial l(y) \ni uBy & \text{a.e. on }]0, T[\\ y(0) = y_0 \end{cases} \tag{1.2'}$$

where

$$l(y) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\nabla y|^2 dx + \int_\Omega j(y)\, dx & \text{if } y \in H_0^1(\Omega) \text{ and } j(y) \in L^1(\Omega) \\ +\infty \text{ otherwise} & \end{cases}$$

and $\partial j = \beta$.

Also we remind (see [1]) that if $y_0 \in D(l)$ and $u \in L^2(0, T)$ then (1.2)′ admits a strong solution y which in addition satisfies $y' \in L^2(0, T; L^2(\Omega))$.

Moreover the operator $\Gamma : L^2(0, T) \to C([0, T]; L^2(\Omega))$ defined by $\Gamma u = y$, where y is the solution of (1.2) corresponding to $u \in L^2(0, T)$ is compact.

This allows us to say that there exists at least on pair (y^*, u^*) (called optimal) that minimizes (1.1) and satisfies (1.2).

In [1] we have established necessary optimality conditions in order that (y^*, u^*) be an optimal pair for problem (1.1), (1.2).

It is the purpose of the present paper to give a method for approximating the optimal pair (y^*, u^*), for the pay-off function with $\psi \equiv 0$.

It uses a Galerkin scheme, regularising techniques and a gradien algorithm.

In order to facilitate the reference to the problem (1.1), (1.2) (with $\psi \equiv 0$) we shall call it problem (P).

In what follows we shall use the notations $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $V' = H^{-1}(\Omega)$.

$(\cdot\,, \cdot)$ denotes the inner products in H and also the duality between V and V', while $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_*$ designate the norms in V, H and V' respectively.

There is no danger of confusion if the same notation is used for the norms in R^n and R, as in H.

2. Galerkin approximation of Problem (P). Firstly we shall describe a finite element aproximation of the spaces V and H. The notations we use are standard (see [2]).

For this let $\mathscr{H}$ be a neighbourhood of 0 in R^n and $h \in \mathscr{H}$, $h \neq 0$ at parameter destinated to converge to 0. For any $h \in \mathscr{H}$ the following elements are given:

i) a finite real linear space V_h;

ii) the prolongation $p_h : V_h \to W_h \subset V$ which is linear, bounded and one-to-one, $W_h = p_h(V_h)$;

iii) the restriction $r_h : H \to V_h$, which is linear and bounded.

As regards p_h and r_h we make the following assumptions:

(2.1) There exists a constant $C > 0$ independent of h such that

$$\|p_h r_h\|_{L(H,V)} \leqslant C, \text{ for any } h \in \mathscr{H}.$$

In what follows we shall use the same letter C for denoting different constants independent of h, which will appear in our estimates or assumptions.

Only in special situations we shall use other notation.

(2.2) $$p_h r_h y \to y \text{ strongly in } V, \text{ for any } y \in V.$$

(2.3) The convergence $u_h \to u$ strongly in $L^2(0, T)$ implies

$$g(u_h) \to g(u) \text{ strongly in } L^1(0, T).$$

(2.4) $$l(p_h r_h y_0) \leqslant C, \text{ for all } h \in \mathscr{H}.$$

In V_h we introduce the scalar product $(.,.)_h$ defined by $(y_h, z_h)_h = (p_h y_h, p_h z_h)$, for all $y_h, z_h \in V_h$ and associated norm $|\cdot|_h$,

$$|y_h|_h = |p_h y_h|, \text{ for all } y_h \in V_h.$$

Besides the last norm we define on V_h the norm $\|\cdot\|_h$, by

$$\|y_h\|_h = \|p_h y_h\|, \text{ for any } y_h \in V_h.$$

Because of the compact inclusion $V \subset H$ it is easy to see that

(2.5) $$|y_h|_h \leqslant C_0 \|y_h\|_h, \text{ for any } y_h \in V_h, (C_0 > 0)$$

If we denote by A the operator $-\Delta$ we may define $A_h : V_h \to V_h$ as follows

$$(A_h y_j, z_h)_h = (A\, p_h y_h, p_h z_h), \text{ for all } y_h, z_h \in V_h.$$

As regards A_h, this keeps the properties of A. In the same way is defined the operator $B_h : V_h \to V_h$. It is easy to see that B_h is linear, bounded and

$$\|B_h\|_{L(V_h, V_h)} \leqslant \|B\|_{L(H,H)}$$

Also we define β_h by

$$\beta_h(y_h) = \beta(p_h y_h), \text{ for } y_h \in V_h.$$

In this way equation (1.2) can be written as:

(2.6) $$\begin{cases} y_h' + A_h y_h + \beta_h(y_h) \ni u_h B_h y_h \\ y_h(0) = r_h y_0 \end{cases}$$

or

(2.6)′ $$\begin{cases} y_h' + \partial l_h(y_h) \ni u_h B_h y_h \\ y_h(0) = \gamma_h y_0 \end{cases}$$

where $l_h(y_h) = l(p_h y_h)$, while (1.1) becomes

$$(P_h) \quad \begin{cases} \text{Minimize} \\ F_h(u_h) = \int_0^T (\varphi_h(\Gamma_h u_h(t)) + g(u_h(t)))\, dt \\ \text{subject to (2.6)′.} \end{cases}$$

It is easy to prove, using the same arguments as in [1], [2] that for every $h \in \mathscr{H}$, $h \neq 0$ the problem (P_h) admits at least one optimal pair (y_h^*, u_h^*). The theorem bellow shows that this pairs converge to an optimal pair of the original problems (P).

THEOREM 2.1. *Let $h \in \mathscr{H}$, $h \neq 0$ and (y_h^*, u_h^*) be an optimal pair for problem (P_h). Under the above assumptions*

$u_h^* \to \tilde{u}^*$ weakly in $L^2(0, T)$

$p_h y_h^* \to \tilde{y}^*$ strongly in $C([0, T]; H)$ where $(\tilde{y}^*, \tilde{u}^*)$ is an optimal pair for problem (P).

Proof. (see also [2]). The proof will be done in 3 steps. Step 1. The convergence of the pair (y_h^*, u_h^*) to $(\tilde{y}^*, \tilde{u}^*)$. Step 2. $(\tilde{y}^*, \tilde{u}^*)$ is an admissible pair for problem (P) i.e. satisfies (1.1), (1.2).

Step 3. $(\tilde{y}^*, \tilde{u}^*)$ is an optimal pair for problem (P). We begin with Step 1. Since (y_h^*, u_h^*) is optimal for (P_h) we may write

$$F_h(u_h^*) \leqslant F_h(u_h), \quad \text{for all } u_h \in L^2(0, T) \tag{2.7}$$

Making in (2.7), $u_h \equiv 0$ and having in mind (1.3) we obtain the boundedness of $\{u_h^*\}$ in $L^2(0, T)$.

Hence there exists $\tilde{u}^* \in L^2(0, T)$ to which u_h^* converges weakly in $L^2(0, T)$. On the other hand (y_h^*, u_h^*) satisfies

$$\begin{cases} y_h^{*\prime} + \partial l_h(y_h^*) \ni u_h^* B_h y_h^* \\ y_h^*(0) = \gamma_h y_0 \end{cases} \tag{2.8}$$

Multiplying (2.8) by y_h^* and integrating over $[0, t]$ we obtain through Gronwall's lemma

$$\|p_h y_h^*\|_{C([0,T];H)} \leqslant C, \text{ for all } h \in \mathscr{H}, \tag{2.9}$$

Here we have also used the inequalities $(A_h y_h, y_h) \geqslant 0$ and $(\beta_h y_h, y_h) \geqslant 0$, for all $h \in \mathscr{H}$.

Now multiplying (2.8) by $y_h^{*\prime}$ and integrating on $[0, t]$ one finds

$$\int_0^t |y_h^{*\prime}(s)|_h^2 ds + l_h(y_h^*)(t)) - l_h(y_h^*)(0)) = \tag{2.10}$$

$$= \int_0^t u_h^*(s)(B_h y_h^*(s), y_h^{*\prime}(s))_h ds.$$

Taking into account (2.4), (2.9) and the fact that l_h is bounded below by an affine function we get

$$\|p_h y_h^{*\prime}\|_{L^2(0,T;H)} \leqslant C, \text{ for all } h \in \mathscr{H}. \tag{2.11}$$

Which implies that $\{p_h y_h^*\}$ is uniformly equi-continuous in $C([0, T]; H)$.

Now coming back to (2.10) we see that $\{l_h(y_h^*(t))\}$ is uniformly bounded on $[0, T]$ and since l is coercive on V, we obtain that $\{p_h y_h^*(t)\}$ is bounded in V, hence compact in H for every $t \in [0, T]$.

Finally the Ascoli-Arzelá theorem may be applied to obtain

(2.12) $p_h y_h^* \to y$ strongly in $C([0, T]; H)$ and weakly star in $L^\infty(0, T; V)$.

Step. 2. We multiply (2.8) by $\partial l_h(y_h^*)$ and integrate the resulting equation over $[0, T]$.

It results

$$l_h(y_h^*(T)) + \int_0^T |\partial l_h(y_h(t))|_h^2 \, dt = l_h(y_h^*(0)) +$$

$$+ \int_0^T u_h^*(t)(B_h y_h^*(t), \partial l_h y_h^*(t))_h \, dt$$

and since $l_h(y_h^*)$ and y_h^* are uniformly bounded on $[0, T]$ we get $\|\partial l_h(y_h^*)\|_{L^2(0,T;H)} \leqslant C$, hence there exists $\xi \in L^2(0, T; H)$ such that

$$\partial l_h(y_h^*) \to \xi \text{ weakly in } L^2(0, T; H). \tag{2.13}$$

Now (2.12), (2.13) and the demi-closedness property of ∂l implies

$$\xi(t) \in \partial l(\tilde{y}^*(t)) \text{ a.e. } t \in]0, T[.$$

As we have already seen $\{p_h y_h^*\}$ is bounded in $L^2(0, T; H)$ so there exists $q \in L^2(0, T; H)$ such that

$$p_h y_h^* \to q \text{ weakly in } L^2(0, T; H). \tag{2.14}$$

Multiplying (2.8) by $z \in L^2(0, T; H)$ and integrating from 0 to T we obtain

$$\int_0^T (y_h^*(t), z(t)) \, dt + \int_0^T (\partial l_h(y_h^*(t)), z(t)) \, dt =$$

$$= \int_0^T u_h^*(t)(B_h y_h^*(t), z(t)) \, dt.$$

Tending to the limit in the last equality and using (2.13) and (2.14) we get

$$q + \partial l(\tilde{y}^*) \ni \tilde{u}^* B \tilde{y}^* \quad \text{a.e. on }]0, T[\tag{2.15}$$

$$q(t) = \tilde{y}^{*\prime}(t) \quad \text{a.e. } t \in]0, T[.$$

which implies

This along with (2.15) shows that $(\tilde{y}^*, \tilde{u}^*)$ is an admissible pair for problem (P).

Step 3. Putting in (2.8) $u_h = u^*$ it follows $p_h y_h = y^*$, and since

$$F_h(u_h^*) \leqslant F_h(u^*) = F(u^*), \text{ for all } h \in \mathscr{H}$$

he following inequality is obtained

$$\int_0^T (\varphi(p_h y_h^*(t)) + g(u_h^*(t))) dt \leqslant \int_0^T (\varphi(y^*(t)) + g(u^*(t))) \, dt$$

Making h tend to 0 in the last inequality we obtain $F(\tilde{u}^*) \leq F(u^*)$ which led to the conclusion that $(\tilde{y}^*, \tilde{u}^*)$ is an optimal pair for (P).

3. Regularization of the finite-dimensional problem. The aim of this section is to solve the finite dimensional problem (P_h) for $h \neq 0$ fixed in $\mathscr{H}$. For this, following the ideas in [2] and [3], we shall regularize Problem (P_h) in order to make it differentiable. Let us denote by $n = n(h)$, the dimension of V_h. For any $\varepsilon > 0$, we consider the regularized problem $(P_{h,\varepsilon})$ as follows:

Minimize:

$$F_h^\varepsilon(u_h) = \int_0^T (\varphi_h^\varepsilon(u_h(t)) + g_\varepsilon(u_h(t)))\, dt \tag{3.1}$$

subject to

$$\begin{cases} y_h' + A_h y_h + \beta_h^\varepsilon(y_h) = u_h B_h y_h & \text{a.e. in }]0, T[\\ y_h(0) = \gamma_h y_0 \end{cases} \tag{3.2}$$

where φ_h^ε, g_ε, β_h^ε are the regularizations of φ_h, g and β_h respectively, defined by

$$\varphi_h^\varepsilon(y_h) = \int_{R^n} \varphi_h(y_h - \varepsilon\theta)\rho_n(\theta)d\theta, \quad \rho_n \text{ — being a } C_0^\infty \text{ — mollifier in } R^n$$

$$g_\varepsilon(u) = \inf\{|u - v|^2/2\varepsilon + g(v);\quad v \in H\}$$

$$\beta_{h,\varepsilon} = \varepsilon^{-1}(I - (I + \varepsilon\beta)^{-1})$$

$$\beta_h^\varepsilon(y_h) = \int_{R^n} \beta_{h,\varepsilon}(y_h - \varepsilon\theta)\rho_n(\theta)d\theta$$

As regards (3.2) this may be written as

$$\begin{cases} y_h' + \partial l_h^\varepsilon(y_h) = u_h B_h y_h & \text{a.e. on }]0, T[\\ y_h(0) = \gamma_h y_0 \end{cases} \tag{3.3}$$

where

$$l_h^\varepsilon(y_h) = l_h(y_h) + \int_0^T j_h^\varepsilon(y_h)(t))\, dt - \int_0^T j_h(y_h(t))\, dt \tag{3.4}$$

$$j_h^\varepsilon(y_h) = \int_{R^n} j_{h,\varepsilon}(y_h - \varepsilon\theta)\rho_n(\theta)\, d\theta \tag{3.5}$$

$$j_{h,\varepsilon}(y_h) = \inf\{|y_h - z_h|^2/2\varepsilon| + j_h(z_h),\ z_h \in V_h\}$$

The following technical lemmas are useful to prove the convergence of Problems $(P_{h,\varepsilon})$.

LEMMA 3.1. *For any $y_h \in D(l_h)$ the following two relations hold*

$$\limsup_{\varepsilon \to 0} l_h^\varepsilon(y_h) \leq l_h(y_h) \tag{3.6}$$

$$\liminf_{\varepsilon \to 0} l_h^\varepsilon(y_h) \geq l_h(y_h) \tag{3.7}$$

for any sequence $\{y_h^\varepsilon\}$ strongly convergent to y_h in $L^2(0, T; V_h)$ when $\varepsilon \to 0$.

Proof. From (3.5) we have

$$j_h^\varepsilon(y_h) - j_h(y_h) \leq \int_{R^n} j_h(y_h)\rho_n(\theta)\, d\theta + \frac{\varepsilon}{2}\int_{R^n} |\theta|^2\rho_n(\theta)d\theta - j_h(y_h) = \frac{\varepsilon}{2}\int_{R^n} |\theta|^2\rho_n(\theta)\, d\theta.$$

This, in conjunction with (3.4) gives $l_h^\varepsilon(y_h) \leq l_h(y_h) + \frac{\varepsilon}{2}\int_{R^n} |\theta|^2\rho_n(\theta)\, d\theta$ which implies (3.6).

Let now $\{y_h^\varepsilon\}$ be a sequence convergent to y_h in $L^2(0, T; V_h)$ as $\varepsilon \to 0$, so we may infer on a subsequence

$$y_h^\varepsilon(t) \to y_h(t) \text{ strongly in } V_h, \text{ a.e. } t \in]0, T[\quad \text{for } \varepsilon \to 0$$

Writting

$$j_h^\varepsilon(y_h^\varepsilon(t)) = \int_{R^n} (j_h^\varepsilon(z_h^\varepsilon(t, \theta)) + |z_h^\varepsilon(t, \theta) - y_h^\varepsilon(t) - \varepsilon\theta|_h^2/2\varepsilon\rho_n(\theta)\, d\theta \tag{3.8}$$

and assuming that $\int_0^T j_h^\varepsilon(y_h^\varepsilon(t))\, dt$ is bounded for ε sufficiently small we find that

$z_h^\varepsilon(t, \theta) - y_h^\varepsilon(t) - \varepsilon\theta \to 0$ strongly in V_h a.e. $t \in]0, T[$, $|\theta| \in [0, 1]$,

hence

$z_h^\varepsilon(t, \theta) \to y_h(t)$ strongly in V_h, a.e. $t \in]0, T[$, $|\theta| \in [0, 1]$, for $\varepsilon \to 0$.

Since l_h is lower semicontinuous, using Fatou's lemma and (3.8) we get

$$\liminf_{\varepsilon \to 0} \int_0^T j_h^\varepsilon(y_h^\varepsilon(t))dt \geq \int_0^T\int_{R^n} j_h(y_h(t))\rho_n(\theta)\, d\theta = \int_0^T j_h(y_h(t))\, dt$$

from which (3.7) is easy obtained.

For $\varepsilon > 0$ we define the operator

$$\Gamma_{h,\varepsilon} : L^2(0, T; V_h) \to L^2(0, T; V_h) \text{ given by } \Gamma_{h,\varepsilon} u_h = z_h$$

where z_h, u_h verify the system

$$\begin{cases} z_h' + \partial l_h^\varepsilon(z_h) = u_h B_h z_h \text{ a.e. on } [0, T] \\ z_h(0) = \gamma_h y_0 \end{cases} \tag{3.9}$$

It is relatively easy to show that $\Gamma_{h,\varepsilon}$ is compact. In addition we can prove that $\Gamma_{h,\varepsilon}$ satisfies the properties described by

LEMMA 3.2. *Let $u_h \in L^2(0, T)$ and $\gamma_h y_0 \in D(l_h)$ be fixed. Then*

$$|\Gamma_{h,\varepsilon} u_h - \Gamma_h u_h|_{C([0,T];V_h)} \leq C \cdot \varepsilon^{1/2}, \text{ for all } \varepsilon > 0 \tag{3.10}$$

$$(\Gamma_{h,\varepsilon} u_h)' \to (\Gamma_h u_h)' \text{ weakly in } L^2(0, T; V_h) \text{ for } \varepsilon \to 0. \tag{3.11}$$

Proof. Let y_h^ε be the solution of (3.3) corresponding to u_h i.e.

$$\begin{cases} y_h^{\varepsilon\prime} + \partial l_h^\varepsilon(y_h^\varepsilon) = u_h B_h y_h^\varepsilon \\ y_h^\varepsilon(0) = \gamma_h y_0. \end{cases} \tag{3.12}$$

Multiplying this by $y_h^{\varepsilon\prime}$ and respectively by $y_h^\varepsilon - \gamma_h y_0$ and integrating from 0 to t we obtain

$$\frac{1}{2}\int_0^t |y_h^{\varepsilon\prime}|_h^2\, ds + l_h^\varepsilon(y_h^\varepsilon(t)) \leq l_h^\varepsilon(\gamma_h y_0) + C\int_0^t |u_h|^2\, |y_h^\varepsilon|_h^2\, ds \tag{3.13}$$

and

$$|y_h^\varepsilon(t) - \gamma_h y_0|_h^2 + \int_0^t l_h^\varepsilon(y_h^\varepsilon)\, ds \leq \int_0^t l_h^\varepsilon(\gamma_h y_0)\, ds + \tag{3.14}$$

$$+ \frac{1}{2}\int_0^t |y_h^\varepsilon - \gamma_h y_0|_h^2\, ds + \frac{1}{2}\int_0^t |B_h y_h^\varepsilon|_h^2 |u_h|^2\, ds$$

Since l_h^ε is uniformly bounded below by an affine function, and

$$l_h^\varepsilon(\gamma_h y_0) \leq l_h(\gamma_h y_0) + \varepsilon T/2$$

we obtain by (3.14) via Gronwall's lemma that $\{y_h^\varepsilon\}$ is uniformly bounded in $C([0, T]; V_h)$ with respect to ε. Coming back to (3.13) we observe that $\{y_h^{\varepsilon\prime}\}$ is uniformly bounded in $L^2(0, T; V_h)$, hence $\{y_h^\varepsilon\}$ is bounded in $W^{1,2}(0, T; V_h)$ which in conjunction with (3.12) gives the boundedness of $\{\partial l_h^\varepsilon(y_h^\varepsilon)\}$ in $L^2(0, T; V_h)$.

Now introducing the inequality (see [3])

$$(\partial l_h^\varepsilon(y_h) - \partial l_h^\lambda(z_h), y_h - z_h)_h \geq -C(\varepsilon + \lambda)(|(\partial l_h^\varepsilon)^0(y_h)|_h^2 +$$
$$+ |(\partial l_h^\lambda)^0(z_h)|_h^2 + 1)$$

in (3.12) we get

$$|y_h^\varepsilon(t) - y_h^\lambda(t)|_h \leq C(\varepsilon + \lambda)^{1/2}, \varepsilon, \lambda > 0, t \in [0, T]. \tag{3.15}$$

From the above relations we deduce that there exist $\tilde{y}_h \in W^{1,2}(0, T; V_h)$ and $q_h \in L^2(0, T; V_h)$ such that for $\varepsilon \to 0$ the following convergences hold

$$y_h^\varepsilon \to \tilde{y}_h \text{ strongly in } C([0, T]; V_h)$$
$$y_h^{\varepsilon\prime} \to \tilde{y}_h' \text{ weakly in } L^2(0, T; V_h)$$
$$\partial l_h^\varepsilon(y_h^\varepsilon) \to q_h \text{ weakly in } L^2(0, T; V_h).$$

This together with (3.12) gives

$$\tilde{y}_h'(t) + q(t) = u_h(t) B_h \tilde{y}_h(t) \text{ a.e. } t \in]0, T[$$

In order to prove (3.11)) we must show that

$$q_h(t) \in \partial l_h(\tilde{y}_h)(t)) \text{ a.e. } t \in]0, T[.$$

For this we multiply (3.12) with $y_h^\varepsilon - y_h$ and integrate from s to t obtaining

$$\frac{|y_h^\varepsilon(t) - y_h|^2 - |y_h^\varepsilon(s) - y_h|^2}{2} + \int_s^t (l_h^\varepsilon(y_h^\varepsilon(\tau)) - l_h^\varepsilon(y_h))\, d\tau$$

$$\leq \int_s^t u_h(\tau)(B_h y_h^\varepsilon(\tau), y_h^\varepsilon(\tau) - y_h)_h\, d\tau \text{ for all } y_h \in V_h$$

and $0 \leq s < t \leq T$.

Making $\varepsilon \to 0$ and using Fatou's lemma we obtain

$$\frac{|\tilde{y}_h(t) - y_h|_h^2 - |\tilde{y}_h(s) - y_h|_h^2}{2} + \int_s^t l_h(\tilde{y}_h(\tau))d\tau - (t - s)\, l_h(y_h) \leq$$

$$\int_s^t u_h(\tau)(B_h \tilde{y}_h(\tau), \tilde{y}_h(\tau) - y_h)_h\, d\tau.$$

Dividing the last by $(t - s)$ and making $s \to t$ we obtain

$$\tilde{y}_h'(t) + \partial l_h(y_h(t)) \ni u_h(t) B_h \tilde{y}_h(t) \text{ a.e. } t \in]0, T[,$$

hence $q_h(t) \in \partial l_h(\tilde{y}_h(t))$ a.e. $t \in]0, T[$.

Finally (3.11) follows from (3.15).

Since the properties invoked for establishing the existence of an optimal pair for problem (P) are kept for $(P_{h,\varepsilon})$ we conclude that there exists an optimal pair in this case.

The main result of this section presents the convergence of the optimal pairs for problems $(P_{h,\varepsilon})$ to those of (P_h), for $\varepsilon \to 0$.

THEOREM 3.1. *Let* $(y_h^{\varepsilon *}, u_h^{\varepsilon *})$ *be an optimal pair for the problem* $(P_{h,\varepsilon})$.

Then, for $\varepsilon \to 0$ *we have*

$$u_h^{\varepsilon *} \to u_h \text{ weakly in } L^2(0, T)$$

$$y_h^{\varepsilon *} \to y_h \text{ strongly in } C([0, T]; V_h)$$

where (y_h, u_h) *is an optimal pair for* (P_h).

Proof. Since $(y_h^\varepsilon, u_h^\varepsilon)$ is an optimal pair for $(P_{h,\varepsilon})$ the following inequality holds

$$\int_0^T (\varphi_h^\varepsilon(y_h^{\varepsilon *}) + g_\varepsilon(u_h^{\varepsilon *}))dt \leqslant \int_0^T (\varphi_h^\varepsilon(\Gamma_{h,\varepsilon} u_h^*) + g_\varepsilon(u_h^*))\,dt$$

where u_h^* is an optimal control for (P_h).

Using (3.10), the lipschitzianity of φ_h and the inequality $g_\varepsilon \leqslant g$ we obtain after some calculations

$$|\varphi_h^\varepsilon(\Gamma_{h,\varepsilon}(u_h^*)) - \varphi_h(\Gamma_h u_h^*)| \leqslant C\,\varepsilon^{1/2}$$

which implies

$$\limsup_{\varepsilon \to 0} \int_0^T (\varphi_h^\varepsilon(y_h^{\varepsilon *}) + g_\varepsilon(u_h^{\varepsilon *})\,dt \leqslant \int_0^T (\varphi_h(\Gamma_h u_h^*) + g(u_h^*))\,dt.$$

Now the coercivity of g_ε implies that $\{u_h^\varepsilon\}$ is bounded in $L^2(0, T)$, hence on a subsequence we have

$$u_h^\varepsilon \to \tilde{u}_h \text{ weakly in } L^2(0, T) \text{ for } \varepsilon \to 0.$$

The rest of the proof goes in an analogous manner as in Lemma 3.2.

In order to solve the problem $(P_{h,\varepsilon})$ by an approximate method we shall use a gradient algorithm. This algorithm is inspired from [2]. For its convergences we refer the reader to [4]. For our purposes we must calculate the derivative of the functional $F_h^\varepsilon : L^2(0, T) \to \bar{R}$ defined by

$$F_h^\varepsilon(u_h) = \int_0^T (\varphi_h^\varepsilon(y_h) + g_\varepsilon(u_h))\,dt \text{ where } y_h = \Gamma_{h,\varepsilon}(u_h)$$

$$(F_h^{\varepsilon\prime}(u_h), v_h) = \lim_{\substack{\lambda \to 0 \\ \lambda > 0}} \frac{F_h^\varepsilon(u_h + \lambda v_h) - F_h^\varepsilon(u_h)}{\lambda} =$$

$$= \int_0^T (\nabla \varphi_h^\varepsilon(y_h), z_h)_h\,dt + \int_0^T (\nabla g_\varepsilon(u_h), v_h)\,dt$$

where (z_h, v_h) satisfies the system

$$\begin{cases} z_h' + \nabla^2 l_h^\varepsilon(y_h) z_h = v_h B_h y_h + u_h B_h z_h \\ z_h(0) = 0. \end{cases}$$

Let us consider now $p_h \in W^{1,2}(0, T; V_h)$ the solution of

$$\begin{cases} p_h' - \nabla^2 l_h^\varepsilon(y_h)\, p_h = -u_h B_h^* p_h + \nabla \varphi_h^\varepsilon(y_h) \\ p_h(T) = 0 \end{cases}$$

After some calculations involving the last formulas we obtain

$$F_h^{\varepsilon\prime}(u_h) = \nabla g_\varepsilon(u_h) - (p_h, B_h y_h)_h$$

and the algorithm, we propose is the following.

Step 0. choose $u_h^{(0)}$

set $n := 0$

Step 1: compute $y_h^{(n)}$ by solving the system

$$\begin{cases} y_h^{(n)\prime} + \nabla l_h^\varepsilon(y_h^{(n)}) = u_h^{(n)} B_h y_h^{(n)} \\ y_h^{(n)}(0) = \gamma_h y_0 \end{cases}$$

Step 2: test if the pair $(y_h^{(n)}, u_h^{(n)})$ is satisfactory

if YES : then STØP

if NOT, GØ TØ Step 3

Step 3: Compute $p_h^{(n)}$ from the system

$$\begin{cases} p_h^{(n)\prime} = \nabla^2 l_h^\varepsilon(y_h^{(n)}) p_h^{(n)} - u_h^{(n)} B_h^* p_h^{(n)} + \nabla \varphi_h^\varepsilon(y_h^{(n)}) \\ p_h^{(n)}(T) = 0 \end{cases}$$

Step 4: Compute $u_h^{(n+1)}$ given by

$$u_h^{(n+1)} = u_h^{(n)} - \rho_n(\nabla g_\varepsilon(u_h^{(n)}) - (p_h^{(n)}, B_h y_h^{(n)})_h)$$

Step 5: Set $n := n + 1$.

GØTØ Step 1.

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