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STRICTLY CONCAVE AND STRICTLY SUPERHARMONIC FUNCTIONS

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The potentials p, defined on a harmonic space X, with the property: if μ and ν are measures on X such that $\int pd\mu = \int pd\nu < \infty$ and $\int^* sd\mu \leqslant \int^* sd\nu$ for any nonnegative hyperharmonic function s, then $\mu = \nu$ are investigated in [3] (see also [4, p. 166] and [8, p. 438]).

This paper is concerned with functions having similar properties.

1. Strictly superharmonic functions. Let D be an open subset of $R^n (n \geqslant 2)$ having a Green function G. Let u be a superharmonic function on D. Then

(1.1) $$u(x) \geqslant L(u : x, \delta)$$

for each $x \in D$ and $\delta > 0$ such that $\bar{B}_{x,\delta} \subset D$. Here $B_{x,\delta}$ is the open ball with center x and radius δ; $L(u : x, \delta)$ is the average of u on the boundary of $B_{x,\delta}$ (relative to surface area).

We say that u is a *strictly superharmonic function on* D (see [1, p. 72]) if (1.1) holds with "$>$".

PROPOSITION 1.1. *Let θ be a measure on D; suppose that $u = G\theta$ is a finite continuous potential on D. Then u is strictly superharmonic on D iff*

(1.2) $\theta(U) > 0$ *for each non-empty open set* $U \subset D$.

Proof. Let U be a non-empty open set, $\theta(U) = 0$. Then u is harmonic on U [7, Th. 6.6].

Conversely, suppose that (1.2) holds. Let $x \in D, \delta > 0, \bar{B}_{x,\delta} \subset D$. u is superharmonic, hence $L(u : x, \delta) \leqslant u(x)$. Suppose $L(u : x, \delta) = u(x)$. Let $B = B_{x,\delta}$ and $h = PI(u, B)$ [7, p. 23]. Then h is harmonic on B, $h \leqslant u$ on $\bar{B}$ [7, Th. 4.11] and $\lim_{y \to z \in \partial B} h(y) = u(z)$ [7, Th. 2.8]. It follows $h(x) = L(u : x, \delta) = u(x)$.

Let $x_1 \in B, h(x_1) < u(x_1)$. Let $c > 0$, $x_1 \in \partial B_{x,c}$. Then $L(h : x, c) < L(u : x, c)$. On the other hand, $h(x) = L(h : x, c)$ and $u(x) = L(u :$

$x, c)$ [7, L.4.18]. Hence $L(h : x, c) = L(u : x, c)$, a contradiction. Thus $u = h$ on B, i.e. u is harmonic on B. But then $\theta(B) = 0$ [7, Th.6.9], which contradicts (1.2).

Let λ be the Lebesgue measure on R^n.

THEOREM 1.1. *Let θ be a measure on D such that*

(1.3) $\lambda(M) = 0$ *for each* $M \subset D$ *with* $\theta(M) = 0$.

Let $u = G\theta$. *If* μ *and* ν *are measures on* D *such that* $\mu(u) = \nu(u) < \infty$ *and* $\mu(s) \leqslant \nu(s)$ *for each nonnegative superharmonic function* s, *then* $\mu = \nu$.

Proof. Let D' be a component of D. Then $G\mu$ is superharmonic on D' or $G\mu = \infty$ on D'. Let $G\mu = \infty$ on D'. From (1.3) and [7, Th.6.14] we deduce $\mu(u) = \mu(G\theta) = \theta(G\mu) = \infty$, a contradiction. Hence $G\mu$ and $G\nu$ are potentials [7, p. 98]. From $\mu(u) = \nu(u)$ it follows that $\mu(G\theta) = \nu(G\theta)$, i.e. $\theta(G\mu) = \theta(G\nu)$ [7, Th. 6.14]. Now $G(x, .)$ is nonnegative superharmonic for each $x \in D$, hence $G\mu(x) = \mu(G(x, .)) \leqslant \nu(G(x,.)) = G\nu(x)$, $x \in D$. We deduce $G\mu = G\nu$ θ-a.e., i.e. $G\mu = G\nu$ λ-a.e. $G\mu$ and $G\nu$ are finite λ-a.e. [7, Th.4.10]. Hence $G\mu$ and $G\nu$ are finite and equal λ-a.e. From the proof of Theorem 6.15 in [7] we deduce that $\mu = \nu$.

Remark 1.1. Let θ be a measure on D. Clearly (1.2) holds if (1.3) holds. Let M be a closed subset of D, $\lambda(M) > 0$, $\operatorname{int}(M) = \emptyset$. Let $\theta(B) = \lambda(B \cap (D \setminus M))$ for each Borel set $B \subset D$. Then (1.2) holds but (1.3) does not hold.

Remark 1.2. The function u from Theorem 1.1 is superharmonic or $=\infty$ on each component of D. Let D' be a component on which u is superharmonic. Using Theorem 1.1 it is easy to prove that u is strictly superharmonic on D'.

2. Pointwise strictly S-concave functions. Geometric characterizations. Let K be a compact metric space. Let S be a subset of $C(K)$ such that

(2.1) $$aS + bS \subset S \text{ for all } a, b \geqslant 0$$

(2.2) There exists $s_0 \in S$, $s_0(x) > 0$ for all $x \in K$

We say that $u \in C(K)$ is *pointwise strictly S-concave* (see [2, p. 60 and p. 40]) if

(2.3) $$[x \in K,\ \mu \in M_+(K),\ \mu \neq e_x,\ \mu(s) \leqslant s(x),\ s \in S] \Rightarrow \mu(u) < u(x)$$

Here $M_+(K)$ is the set of all positive Radon measures on K and e_x is the Dirac measure at x.

Example 2.1. Let K be a compact convex metrizable subset of a locally convex Hausdorff space. Let $S = \{\min(h_1, \ldots, h_n) : n \in N,\ h_i$ continuous and affine on $K\}$. Then $u \in C(K)$ is pointwise strictly S-concave iff u is strictly concave on K (see [9]).

Example 2.2. Let X be a strictly harmonic space [1] and K a compact subset of X. Let s be the function from the proof of Theorem 2.7.6 in [1]. Then $s|K$ is pointwise strictly S-concave, where $S = ({}_+\mathcal{S}_X \cap C(X))|K$.

Let us remark that in the above examples the cones S are min-stable.

In the proof of the next result (which, in case $\pm 1 \in S$ is contained in Corollary 1 from [5]) the hypothesis that K is metrizable is not necessary.

LEMMA 2.1. *If S is min-stable and $g \in C(K)$, then the following assertions are equivalent:*

(2.4) $$[x \in K,\ \mu \in M_+(K),\ \mu(s) \leqslant s(x)\ \forall s \in S] \Rightarrow \mu(g) \leqslant g(x)$$

(2.5) $$g \in \bar{S}$$

(2.6) $$\forall x \in K\ \forall \varepsilon > 0\ \exists s \in S : s \geqslant g,\ s(x) < g(x) + \varepsilon.$$

Proof. (2.4)$\Rightarrow$(2.5). S being min-stable, we apply the Choquet-Deny type theorem established in [6, Ex.3.2]. Thus, if $g \notin \bar{S}$ then there exists a lattice homomorphism $\delta : C(K) \to R$ and a measure $\nu \in M_+(K)$ such that $\nu(s) \leqslant \delta(s)$ for each $s \in S$ and $\nu(g) > \delta(g)$. If $\delta = 0$, then $0 \leqslant \nu(s_0) \leqslant \delta(s_0) = 0$, hence $\nu = 0$; it follows $0 = \nu(g) > \delta(g) = 0$. Hence $\delta \neq 0$. Then there exist $r > 0$ and $x \in K$ such that $\delta = re_x$. If $\mu = (1/r)\nu$, then $\mu(s) \leqslant s(x)$ for each $s \in S$.

(2.4) implies $\mu(g) \leqslant g(x)$ and therefore $\nu(g) \leqslant \delta(g)$, a contradiction.

(2.5) $\Rightarrow$(2.6). Let $x \in K$ and $\varepsilon > 0$. Let $d, q \in R$, $0 < d < qs_0 < \varepsilon/2$. Let $s_1 \in S$, $\|g - s_1\| < d$. Then $s = s_1 + qs_0$ satisfies (2.6).

(2.6) $\Rightarrow$(2.4). Let x and μ be as in (2.4). Let $\varepsilon > 0$ and let $s \in S$, $s \geqslant g$, $s(x) < g(x) + \varepsilon$. Then $\mu(g) < g(x) + \varepsilon$. Since ε was arbitrary it follows $\mu(g) \leqslant g(x)$.

Remark 2.1. The min-stability of S was used only in the proof of the implication (2.4)$\Rightarrow$(2.5). Another proof of this implication may be obtained applying Corollary 1.6 and Theorem 1.11 from [2].

The concave and strictly concave functions in $C[a, b]$ may be characterized geometrically by studying the position of the tangent line at any point of the graph. We shall obtain similar characterizations for the pointwise strictly S-concave functions.

(2.4) means that g is pointwise S-concave (see [2, p. 40]). From (2.4)$\Leftrightarrow$(2.5) we deduce that if S is min-stable, then the set of all pointwise S-concave functions coincides with $\bar{S}$. The equivalence (2.4)$\Leftrightarrow$(2.6) yields a geometric characterization of the pointwise S-concave functions.

The next proposition gives a sufficient condition in order that a function be pointwise S-concave.

PROPOSITION 2.1. *Let* $g \in C(K)$. *If*

(2.7) $\forall x \in K\ \exists s \in S : s \geqslant g,\ s(x) = g(x)$,

then g is pointwise S-concave.

Proof. (2.7)$\Rightarrow$(2.6)$\Rightarrow$(2.4) (see also Remark 2.1).

A similar sufficient condition for the pointwise strictly S-concavity is given in

Proposition 2.2. *Suppose that for each* $x \in K$ *there exists* $s_x \in S$ *such that* $s_x(x) \to 0$. *Let* $f \in C(K)$. *If*

$$\forall x \in K\ \exists s \in S : s(y) > f(y)\ \forall y \neq x \text{ and } s(x) = f(s), \tag{2.8}$$

then f *is pointwise strictly* S*-concave.*

Proof. Let $x \in K$, $\mu \in M_+(K)$, $\mu \neq e_x$, $\mu(s) \leqslant s(x)$, $s \in S$. Let $s \in S$ with $s(y) > f(y)$ for all $y \neq x$ and $s(x) = f(x)$. Then $\mu(f) \leqslant \mu(s) \leqslant s(x) \leqslant f(x)$. If $\mu(f) = f(x)$, then $\mu(f) = \mu(s)$, i.e. $\operatorname{supp}(\mu) \subset \{x\}$. We deduce $\mu = ae_x$, with $a \geqslant 0$. From $\mu(s_0) \leqslant s_0(x)$ and $\mu(s_x) \leqslant s_x(x)$ it follows that $a \leqslant 1$, $a \geqslant 1$, hence $\mu = e_x$, a contradiction. Therefore $\mu(f) < f(x)$.

A necessary and sufficient condition for the pointwise strictly S-concavity is given in

Theorem 2.1. *A function* $f \in C(K)$ *is pointwise strictly* S*-concave iff* $\forall x \in K$, $\alpha < 0$, $\beta > 1$, V *open neighborhood of* x, $\exists s \in S$, $c > 0$:

$$(cf - s)(x) = 1 \tag{2.9}$$

$$cf - s \leqslant \beta \text{ on } K \tag{2.10}$$

$$cf - s \leqslant \alpha \text{ on } K \setminus V. \tag{2.11}$$

Proof. (see also [12]). Let $T = \{-cf + s : c > 0, s \in S\}$. Then f is pointwise strictly S-concave iff

$$\{x \in K : [\mu \in M_+(K), \mu(t) \leqslant t(x), t \in T] \Rightarrow \mu = e_x\} = K. \tag{2.12}$$

On the other hand, by Corollary 3.6 in [13], (2.12) is equivalent to $\forall\ x \in K$, $\alpha < 0$, $\beta > 1$, V open neighborhood of x, $\exists t \in T$:

$$t(x) < 0,\ t \geqslant \beta t(x) \text{ on } K,\ t \geqslant \alpha t(x) \text{ on } K \setminus V.$$

But $t = -c'f + s'$, with $c' > 0$, $s' \in S$. It suffices to consider $s = -s'/t(x)$, $c = -c'/t(x)$.

Remark 2.2. Proposition 2.2 may be obtained as a consequence of Theorem 2.1. Indeed, let $x \in K$, $\alpha < 0$, $\beta > 1$, V open neighborhood of x. Let $s' \in S$, $s'(y) > f(y)$ for all $y \neq x$, $s'(x) = f(x)$. Let $s_1 \in S$, $s_1(x) = -1$ and let $s = cs' + s_1$. If $c > 0$, then $s \in S$. We have $cf - s = c(f - s') - s_1$, hence $(cf - s)(x) = 1$ for all $c > 0$. For a sufficiently large c, (2.10) and (2.11) hold too.

3. Strictly S-concave functions. We say that $u \in C(K)$ is *strictly* S*-concave* (see [2, p. 22]) if

$$[\mu, \nu \in M_+(K), \mu \neq \nu, \mu(s) \leqslant \nu(s), s \in S] \Rightarrow \mu(u) < \nu(u). \tag{3.1}$$

Clearly, every strictly S-concave function is pointwise strictly S-concave.

Theorem 3.1. *If* S *is min-stable, then every pointwise strictly* S*-concave function is strictly* S*-concave.*

Proof. Let f be pointwise strictly S-concave. Then (2.4) holds, hence $f \in \bar{S}$. Let $\mu, \nu \in M_+(K)$, $\mu(s) \leqslant \nu(s)$, $s \in S$. Then $\mu(f) \leqslant \nu(f)$. Suppose $\mu(f) = \nu(f)$; we have to prove that $\mu = \nu$.

Let $t_n \in S$, $t_n \to f$. Since K is metrizable, $C(K)$ is separable, hence there exist $s_n \in S$ with $S \subset \operatorname{cl}\{s_n : n \in N\}$.

Applying Theorem 1.12 in [2] we deduce that there exists a map $x \to T_x$ of K into $M_+(K)$ such that

(3.2) For every $g \in C(K)$ the function $x \to T_x(g)$ is ν-summable and $\mu(g) = \int_K T_x(g)\,d\nu(x)$

and

(3.3) For every $s \in S$ we have $T_x(s) \leqslant s(x)$ ν-a.e. on K.

Then there exists $A \subset K$, $\nu(A) = 0$ such that for all $x \in K \setminus A$ and all $n \in N$:

$$T_x(t_n) \leqslant t_n(x) \tag{3.4}$$

and

$$T_x(s_n) \leqslant s_n(x) \tag{3.5}$$

From (3.4) it follows that $T_x(f) \leqslant f(x)$ for all $x \in K \setminus A$.

On the other hand, $\nu(f) = \mu(f) = \int_K T_x(f)\,d\nu(x)$; hence

$$T_x(f) = f(x)\ \nu\text{-a.e.} \tag{3.6}$$

From (3.5) we deduce $T_x(s) \leqslant s(x)$ for all $x \in K \setminus A$ and all $s \in S$. It follows that there exists $B \subset K$, $\nu(B) = 0$ such that

(3.7) $T_x(f) = f(x)$ and $T_x(s) \leqslant s(x)$ for all $x \in K \setminus B$ and all $s \in S$.

But f is pointwise strictly S-concave; it follows that $T_x = e_x$ for all $x \in K \setminus B$. Then, for each $g \in C(K)$ we have

$$\mu(g) = \int_K T_x(g)\,d\nu(x) = \int_K g(x)\,d\nu(x) = \nu(g).$$

and the theorem is proved.

Some applications of this result are given in [10], [11]. Here we present another application.

With the notations from Example 2.2, let $\mu \in M_+(K)$, $E \subset K$, μ^E the measure defined in [1, Th. 3.4.1]. Then $\mu^E \in M_+(K)$ [1, Th. 3.4.3].

Proposition 3.1. *If* $\mu(C_K E) > 0$, *then* $\mu^E(p) < \mu(p)$ *for any strict potential* p *on* X.

Proof. See [1, p. 116].

Proposition 3.2. *If* $\mu^E \neq \mu$, *then* $\mu^E(f) < \mu(f)$ *for each pointwise strictly* S*-concave function* f.

Proof. S is min-stable [1, Cor. 1.1.2]. By Theorem 3.1, f is strictly S-concave. We have $\mu^E(s) \leqslant \mu(s)$ for all $s \in S$ [1, p. 50 and Cor. 3.4.2]. Hence $\mu^E(f) < \mu(f)$.

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