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A NEW ALGORITHM FOR PERIODIC SPLINE SOLUTIONS TO SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. A new method which provides a periodic spline approximation to a periodic solution of

$$\ddot{x} + k_1 x + k_2 x^n f(t) = 0, \; k_2 \ll 1$$

is presented. The method can be applied to differential equations of any order. An attempt has been made using spline functions of continuity class C^{m-2} rather than the shooting method to ensure the periodicity of such a solution. Some computational examples are given and numerical results indicate the efficiency of the procedure.

1. Introduction. Here, we shall be concerned with the quasi-linear differential equation

$$(1.1) \qquad \ddot{x} = F(t, x) = -k_1 x - k_2 x^n f(t), \; \dot{x}(t_0) = x_0, \; x(t_0) = x_0$$

which governs some physical system and has always been an important problem for scientists and engineers.

Our aim is to obtain periodic solutions to (1.1), whenever it exists as the limit of a sequence of spline functions of degree m and continuity class C^{m-2}.

Let us assume that the following conditions are valid:
(*i*) $F \varepsilon C^{m-1}$ in some domain $D = \{(t, x) | a = t_0 \leqslant t \leqslant b\}$ and which satisfies a Lipschitz condition with respect to x, i.e.,

$$|F(t, x_1) - F(t, x_2)| \leqslant L |x_1 - x_2|.$$

(*ii*) $f(t)$ is sufficiently smooth periodic function with period $T = b - t_0$ and n is a non negative integer.

It appears that this method offers significant advantages over other methods. Special cases and other procedures have been examined in [4] and [2] which use the so-called averaging method. Also in [1] a cubic spline method to obtain periodic solutions to second order diferential difference equations is investigated. Moreover, an approach (see [3]) to solve highly oscillatory ordinary differential equations is dealt with there.

2. Construction of the Approximating Spline Function. Let $x(t)$ be the exact solution of (1.1). We divide $[a, b]$ into subintervals $I_\nu = [t_0 + 2(\nu - 1)h, 2\nu h]$, $\nu = 1(1)N$ of equal length $H = 2h$, $h = (b - a)/2N$, $N > m$.

Now for $t \in I_\nu$ define the spline function $S(t)$, of degree m, $m \geqslant 4$, and in the continuity class $C^{m-1}[t_0, b]$, as an approximation to (1.1) by

$$S(t) = \sum_{j=0}^{m} \frac{S^{(j)}_{2(i-1)}}{j!} [t - t_0 - 2(i-1)h]^j + \sum_{j=m-1}^{m} \frac{C_{j,i-1}}{j!} [t - t_0 - 2(i-1)h]^j \tag{2.1}$$

where the parameters $C_{j,i-1}$ are determined according to the relations

$$S(t_0 + lh) = -k_1 S(t_0 + lh) - k_2 S^n(t_0 + lh) f(t_0 + lh), \tag{2.2}$$

$$l = 2i - 1,\ 2i;\ i = 1(1)N,$$

and

$$S^{(j)}_{2\nu} = S^{(j)}(t_0 + 2\nu h);\quad S^{(j)}_0(t_0) = x^{(j)}_0$$

We show that the above construction defines $S(t)$ uniquely as a spline o degree m, deficiency 2 in the class $C^{m-2}[a, b]$ for fixed k_1 and k_2.

Writing (2.1) as:

$$S(t) = A_{i-1}(t) + \sum_{j=m-1}^{m} \frac{C_{j,i-1}}{j!} [t - t_0 - 2(i-1)h]^j \tag{2.1'}$$

it is clear that each $A_{i-1}(t)$ is uniquely determined by virtue of the continuity condition of spline.

Applying conditions (2.2) to (2.1′), we have

$$C_{m-1,i-1} = \Phi_{i-1,1}(C_{m-1,i-1}, \tilde{C}_{m,i-1}; h)$$

$$\tilde{C}_{m,i-1} = \Phi_{i-1,2}(C_{m-1,i-1}, \tilde{C}_{m,i-1}; h);\quad \tilde{C}_{m,i-1} = hC_{m,i-1}$$

which can be written in the compact form

$$\underline{C}_{i-1} = \Phi_{i-1}(\underline{C}_{i-1}; h)$$

where

$$C_{m-1,i-1} = \frac{-2(m-3)!}{h^{m-3}} \Big\{ k_1 \Big[A_{i-1}(t_0 + (2i-1)h + \frac{h^{m-1}}{(m-1)!} (C_{m-1,i-1} + \frac{1}{m} \tilde{C}_{m,i-1}) \Big] + k_2 \Big[A_{i-1}(t_0 + (2i-1)h) + \frac{h^{m-1}}{(m-1)!} (C_{m-1,i-1} + \frac{1}{m} \tilde{C}_{m,i-1}) \Big]^n \cdot f(t_0 + (2i-1)h) + \ddot{A}_{i-1}(t_0 + (2i-1)h) \Big\} + \frac{(m-3)!}{(2h)^{m-3}} \Big\{ k_1 \Big[A_{i-1}(t_0 + 2ih) + \frac{(2h)^{m-1}}{(m-1)!} (C_{m-1,i-1} + \frac{2}{m} \tilde{C}_{m,i-1}) \Big] + k_2 \Big[[A_{i-1}(t_0 + 2ih) + \frac{(2h)^{m-1}}{(m-1!} \Big(\tilde{C}_{m-1,i-1} + \frac{2}{m} C_{m,i-1} \Big) \Big]^n f(t_0 + 2ih) + \ddot{A}_{i-1}(t_0 + 2ih) \Big\}$$

and

$$\tilde{C}_{m,i-1} = \frac{-(m-2)!}{(2h)^{m-3}} \Big\{ k_1 \Big[A_{i-1}(t_0 + 2ih) + \frac{(2h)^{m-1}}{(m-1)!} \Big(C_{m-1,i-1} + \frac{2}{m} \tilde{C}_{m,i-1} \Big) \Big] + k_2 \Big[A_{i-1}(t_0 + 2ih) + \frac{(2h)^{m-1}}{(m-1)!} \Big(C_{m-1,i-1} + \frac{2}{m} \tilde{C}_{m,i-1} \Big) \Big]^n f(t_0 + 2ih) + \ddot{A}_{i-1}(t_0 + 2ih) \Big\} + \frac{(m-2)!}{h^{m-3}} \Big\{ k_1 \Big[A_{i-1}(t_0 + (2i-1)h) + \frac{h^{m-1}}{(m-1)!} \Big(C_{m-1,i-1} + \frac{1}{m} \tilde{C}_{m,i-1} \Big) \Big] + k_2 \Big[A_{i-1}(t_0 + (2i-1)h) + \frac{h^{m-1}}{(m-1)!} \Big(C_{m-1,i-1} + \frac{1}{m} \tilde{C}_{m,i-1} \Big) \Big]^n f(t_0 + (2i-h)h + \ddot{A}_{i-1}(t_0 + (2i-h) \Big\}.$$

Let $\underline{C}^1_{i-1}, \underline{C}^2_{i-1} \in R^2$ and

$$\|\underline{C}^1_{i-1} - \underline{C}^2_{i-1}\| = |C^1_{m-1,i-1} - C^2_{m-1,i-1}| + |\tilde{C}^1_{m,i-1} - \tilde{C}^2_{m,i-1}|$$

Thus, in consideration of condition (*i*), a straightforward calculation shows that

$$\|\Phi_{i-1}(\underline{C}^1_{i-1}) - \Phi_{i-1}(\underline{C}^2_{i-1})\| \leqslant \frac{Lh^2(5m-4)}{(m-1)(m-2)} \|\underline{C}^1_{i-1} - \underline{C}^2_{i-1}\|.$$

Clearly the vector-valued function Φ_{i-1} defines a strong contraction mapping for all h satisfying $\frac{Lh^2(5m-4)}{(m-1)(m-2)} < 1$.

Hence $\underline{C}_{i-1}$ are uniquely determined and may be found by the iteration

$$\underline{C}^{(p+1)}_{i-1} = \Phi_{i-1}(\underline{C}^{(p)}_{i-1}; h);\quad p = 0, 1, \ldots\ldots$$

It might be noted that the initial values $C^{(0)}_{m-1,i-1}$ and $C^{(0)}_{m,i-1}$, $i = 1(1)N$, are defined by

$$C^{(0)}_{m-1,i-1} = S^{(m-1)}_{2(i-1)};\quad C^{(0)}_{m,i-1} = S^{(m)}_{2(i-1)}$$

Thus, we have established the following:

THEOREM 1. *If* $\frac{Lh^2(5m-4)}{(m-1)(m-2)} < 1$, *then the spline function* *approximation $S(t)$ to the solution of (1.1), defined by the above construction, exists and is unique.*

3. The Basic Algorithm. Since we seek continuous periodic solution to (1.1) of period $(b - a)$. This solution can be approximated using the spline function constructed above such that

$$(3.1)\qquad \begin{cases} f_1(k_1, k_2) = S(b, k_1, k_2) - x_0 = 0 \\ f_2(k_1, k_2) = S(b, k_1, k_2) - x_0 = 0 \end{cases}$$

An iterative method will be introduced to solve (3.1) for k_1 and k_2 and consequently produces a sequence of splines which converges if the system (3.1) does, then the limit spline will have properties that we require.

The steepest descent method is the proposed iterative method we shall use to find k_1, k_2 that uses a damping factor to avoid the almost singularity or the ill-conditioning of the Jacobian. We can write (3.1) in the form

$$(3.2)\qquad \underline{f}(\underline{k}) = 0$$

where $\underline{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $\underline{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$, and hence

$$\underline{k}^{(p+1)} = \underline{k}^{(p)} - \mu_p w_p' \underline{f}^{(p)}, \quad p = 0, 1, 2, \ldots$$

where

$$\mu_p = \frac{(\underline{f}^{(p)}, w_p w_p' \underline{f}^{(p)})}{\| w_p w_p' \underline{f}^{(p)} \|^2}$$

is the damping factor, $0 < \mu_p \leqslant 1$ and $w_p = \left[\frac{\partial f_i^{(p)}}{\partial k_j} \right]$ is the Jacobian, w_p' its transpose.

To compute the entries of w_p we can represent the derivatives by using the finite difference approximations, e.g.,

$$\frac{\partial f_1}{\partial k_1} = \frac{\partial S(b, k_1, k_2)}{\partial k_1} = \frac{S(b, k_1 + l, k_2) - S(b, k_1, k_2)}{l}$$

where l is a suitable small value which keeps the local error bounded, then we have

$$(3.3)\qquad \begin{cases} k_1^{(p+1)} = k_1^{(p)} - \mu_p \left(f_1^{(p)} \dfrac{\partial f_1}{\partial k_1} + f_2^{(p)} \dfrac{\partial f_2^{(p)}}{\partial k_1} \right) \\ k_2^{(p+1)} = k_2^{(p)} - \mu_p \left(f_1^{(p)} \dfrac{\partial f_1^{(p)}}{\partial k_2} + f_1^{(p)} \dfrac{\partial f_2^{(p)}}{\partial k_2} \right) \end{cases}$$

The limit value of $k^{(p)}$ as $p \to \infty$ will produce a contin periodic solution to (1.1) with the period $T = b - t_0$.

COROLLARY 3.1. *If the system (3.1) has a unique solution given via the iteration (3.3) with suitable initial value $\underline{k}^{(0)}$. Then (3.3) will produce a sequence of splines $\{S^{(p)}(t)\}_{p=0}^{\infty}$ which converges uniformly on $[t_0, b]$.*

It should be pointed out that (see [5], [6]) the spline function approximations $S \in C^{m-2}$, applied so far, are unstable and hence divergent as $h \to 0$ for $m \geqslant 6$ and are stable for $m = 5$.

It should be mentioned that we have not yet been able to find a sufficient condition to assure the convergence of the coupled equation (3.1). A theoritical analysis to determine k is still required.

4. Illustrative Computations. This section decribes results of applying the procedure to some test problem. All computations were fone on an PERKN EL MER 1625 in single precision.

The example is:

$$\ddot{x} + k_1 x + k_2 x^2 \cos t = 0, \quad x(0) = 0.5, \quad \dot{x}(0) = 0$$

Numerical results are listed in Table 4.1 using a spline function of order 4. It turns out that the coefficients $C_{m-1,i-1}$ and $C_{m,i-1}$ remain bounded, as $h \to 0$.

Table 4.1

($m = 4$, $k = 0.999$, $k = 0.01$, $t \in [0, 2]$)

t	$h = \pi/40$			$h = \pi/80$		
	$C_{m-1,i-1}$	$C_{m,i-1}$	$S(t)$	$C_{m-1,i-1}$	$C_{m,i-1}$	$S(t)$
$\pi/2$	0.500377	−0.029388	−0.001515	0.499880	−0.010268	−0.001123
π	−0.001576	−0.509585	−0.500197	−0.001863	−0.511580	−0.500077
$3\pi/2$	−0.500710	0.036228	0.003265	−0.500049	0.017315	0.003270
2π	0.003716	0.509572	−0.500340	0.004384	0.511501	0.500147

It is worth pointing out that the method is unstable and hence divergent if $m \geqslant 6$, as indicated in Table 4.2.

Table 4.2

($m = 6$, $h = \pi/20$, $k_1 = 0.998$, $k_2 = 0.01$)

t	$C_{m-1,i-1}$	$C_{m,i-1}$	$S(t)$
0.31416	−0.801944E+00	6.828334E+00	0.459608E+00
1.25664	−0.374151E+04	0.351783E+05	0.224893E+02
2.51327	−0.183616E+07	0.172662E+08	0.110357E+05
3.14159	−0.114210E+09	0.107385E+10	0.686343E+06

The solution by the spline method is compared to the solution by the generalized averaging method [2] in Table 4.3. Using a spline of order five it is apparent that the spline method can achieve much gain in efficiency ever the generalized averaging method if smaller step size is used (c.f. Table 4.4)

Table 4.3

($m = 5$, $k_1 = 0.9909690$, $k_2 = .0250861$, $h = \pi/40$)

t	solution by the gen. Aver. method	solution by spline method	Difference
$\pi/4$	0.3525831E+00	0.3534291E+00	0.8460283E−03
$\pi/2$	−0.3187807E−06	−0.2067024E−04	0.2035156E−04
$3\pi/4$	−0.3525837E+00	−0.3534939E+00	0.9101629E−03
π	−0.4990201E+00	−0.5003836E+00	0.1363575E−02
$5\pi/4$	−0.3525826E+00	−0.3535237E+00	0.9410977E−03
$3\pi/2$	0.9562263E−06	0.2616155E−03	0.2606594E−03
$7\pi/4$	0.3525841E+00	0.3539488E+00	0.1364648E−02
2π	0.4990201E+00	0.5008035E+00	0.1783490E−02

Table 4.4.

($m = 5$, $k_1 = 0.250024$, $k_2 = 0.010408$, $h = \pi/40$)

t	solution by the gen Aver. method	solution by spline method	Difference
$\pi/2$	0.3547096E+00	0.3511344E+00	0.3079414E−02
π	0.2139158E−02	−0.4748825E−02	0.6887983E−02
$3\pi/2$	−0.3523973E+00	−0.3575117E+00	0.5114376E−02
2π	−0.4992483E+00	−0.5013575E+00	0.2109170E−02
$5\pi/2$	−0.3523962E+00	−0.3560276E+00	0.3631711E−02
3π	−0.3431366E−02	0.2140437E−02	0.5571804E−02
$7\pi/2$	0.3515110E+00	0.3547107E+00	0.3199696E−02
4π	0.4999864E+00	0.5007516E+00	0.7652640E−03

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