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THE ASYMPTOTIC RELATIONS FOR THE INDEFINITELY INCREASING ZEROS OF POLYNOMIALS WITH ALTERNATE COEFFICIENTS

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The author puts the question : which of the n roots of an algebraic real equation increases indefinitely in absolute value when the coefficient a_0 of x^n tends to zero, and gives the answer to this question for such an equation with alternate coefficients provided that all the coefficients of the equation are different from zero; he shows in what manner a certain root of the equation under consideration tends to $+\infty$.

As an asymptotic expression for a polynomial with alternate coefficients of degree $n(n \geq 3)$ we take the following trinomial

(1) $$a_0x^n - a_1x^{n-1} + a_2x^{n-2} \sim \sum_{\nu=0}^{n} (-1)^\nu a_\nu x^{n-\nu}, \; x \to +\infty.$$

With regard to this trinomial we consider a finite sequence of polynomials of the form

(2) $$y_p(x; a_0) = x \cdot y_{p-1}(x, a_0) + (-1)^p a_p, \; p = 3, 4, 5, \ldots, n,$$

with the initial term $y_2(x, a_0) = a_0x^2 - a_1x + a_2$.

We assume the coefficients $a_\nu(\nu = 0, 1, 2, 3, \ldots, p)$ of the polynomial (2) to be positive. The greatest real zero of the polynomial (2) we call the last zero and we denote it by $r_{p,p}$.

It is known from the theory of algebraic equations [2], [3], [1]: If in the equation

(3) $$\sum_{\nu=0}^{p} (-1)^\nu a_\nu x^{p-\nu} = 0, \; (a_\nu > 0)$$

the coefficient a_0 of the highest power tends to zero, then one of the roots of the equation (3) increases indefinitely in absolute value.

In the case of alternate coefficients this theorem will here be proved by means of a new method, that is, by means of certain asymptotic relations.

Referring to this theorem we put the question : which of the roots of equation (3) increases indefinitely in absolute value when the coefficient a_0 tends to zero?

For equation (3) with alternate coefficients we shall prove the following theorems:

1. *The geatest real root of equation* (3), *whose coefficients are all alternate, tends to* $+\infty$ *as* a_0 *tends to zero*

$$r_{p,p}(a_0) \to +\infty,\ a_0 \to +0. \tag{4}$$

2. *In case all the roots of equation* (3) *with alternate coefficients are complex, two conjugate complex roots with the greatest real and positive part become real and the greater one of these real roots tends then to* $+\infty$ *as* a_0 *tends to zero.*

We consider one more finite sequence of equations

$$\sum_{\nu=0}^{p} (-1)^{\nu} a_{\nu} x^{p-\nu} = 0,\ p = 2, 3, 4, \ldots, n\ (a_{\nu} > 0). \tag{5}$$

For equations (5) we shall prove the following theorems:

3. *The roots* $r_{p,p}(p = 2, 3, 4, \ldots, n)$, *which tend to* $+\infty$, *are distributed as follows*

$$0 < r_{22} < r_{44} < \ldots < r_{2p,2p} < r_{2p+1,2p+1} < \ldots < r_{55} < r_{33}. \tag{6}$$

4. *Between the roots* $r_{2p,2p}$ *and* $r_{2p+1,\,2p+1}$ *there exists the asymptotic relation*

$$r_{2p,\,2p}(a_0) \sim r_{2p+1,2p+1}(a_0),\ a_0 \to +0. \tag{7}$$

5. *From* (6) *and* (7) *it follows that the roots, tending to* $+\infty$, *of equations* (5) *are situated in the interval* (r_{22}, r_{33}) *which tends to zero as* a_0 *tends to zero.*

1. We consider a family of parabolas of degree $(n+2)$ with alternate coefficients in its equation

$$y_{n+2}(x, a_0) = \sum_{\nu=0}^{n+2} (-1)^{\nu} a_{\nu} x^{n+2-\nu},\ (a_{\nu} > 0) \tag{8}$$

where a_0 is the parameter of the family. For polynomial (8) we introduce the asymptotic relation

$$y_{n+2}(x, a_0) \sim x^n \cdot (a_0 x^2 - a_1 x + a_2),\ x \to +\infty \tag{9}$$

as well as for the first and the second derivate

$$y'_{n+2}(x, a_0) \sim \left[\binom{n+2}{1} a_0 x^2 - \binom{n+1}{1} a_1 x + \binom{n}{1} a_2\right] \cdot x^{n-1},\ x \to +\infty \tag{10}$$

$$y''_{n+2}(x, a_0) \sim \left[\binom{n+2}{2} a_0 x^2 - \binom{n+1}{2} a_1 x + \binom{n}{2} a_2\right] \cdot 2x^{n-2},\quad x \to +\infty. \tag{11}$$

With regard to relation (9) we consider the sequence of functions

$${}_{n+2}y_2(x, a_0) = x^n \cdot (a_0 x^2 - a_1 x + a_2),\ n = 1, 2, 3, \ldots \tag{12}$$

With regard to formula (2) we consider the parabolas

$$y_2(x, \lambda) = \frac{a_1^2}{4a_2\lambda} x^2 - a_1 x + a_2,\ a_0 = \frac{a_1^2}{4a_2\lambda},\ \lambda > 1 \tag{2_1}$$

cutting the X-axis. For this reason we introduce the parameter λ in place of a_0. By X_2 we denote the X-axis of parabola (2_1). We call this system of coordinates the system (X_2). If, by (2), we multiply $y_2(x, \lambda)$ by x and from this subtract a_3, we obtain

$$y_3(x, \lambda) = \frac{a_1^2}{4a_2\lambda} x^3 - a_1 x^2 + a_2 x - a_3. \tag{2_2}$$

Parabola (2_2) is referred to the system (X_3); we obtain the axis X_3 by putting, in the system (X_2), the parallel to the axis X_2 above this axis at the distance a_3. We obtain the axis X_4 by putting, in the system (X_3), the parallel to the axis X_3 below this axis at the distance a_4; and in a corresponding manner also for the following axes. That is the sence of formula (2).

For the real roots but different from zero of the equation

$${}_{n+2}y_2(x, a_0) = 0,\qquad \text{where } a_0 = \frac{a_1^2}{4a_2\lambda},\ \lambda \geqslant 1, \text{ we obtain}$$

$$r_{21}(\lambda) = \frac{2a_2\lambda}{a_1}\left(1 - \sqrt{1 - \frac{1}{\lambda}}\right),\quad \lambda \geqslant 1, \tag{13}$$

$$r_{22}(\lambda) = \frac{2a_2\lambda}{a_1}\left(1 + \sqrt{1 - \frac{1}{\lambda}}\right),\ \lambda \geqslant 1 \tag{14}$$

$$r_{21}(\lambda) \to \frac{a_2}{a_1},\ \lambda \to +\infty \tag{15}$$

$$r_{22}(\lambda) \to +\infty,\ \lambda \to +\infty \tag{16}$$

$$a_0 \cdot r_{22}(a_0) \to a_1,\ a_0 \to +0. \tag{17}$$

For the equation ${}_{n+2}y'_2(x, a_0) = 0$ we obtain

$$r'_{21}(\lambda, n) = \frac{2a_2\lambda}{a_1} \cdot \frac{n+1}{n+2}\left(1 - \sqrt{1 - \frac{n^2+2n}{(n+1)^2} \cdot \frac{1}{\lambda}}\right) \lambda \geqslant 1 \tag{18}$$

$$r'_{22}(\lambda, n) = \frac{2a_2\lambda}{a_1} \cdot \frac{n+1}{n+2}\left(1 + \sqrt{1 - \frac{n^2+2n}{(n+1)^2} \cdot \frac{1}{\lambda}}\right),\ \lambda \geqslant 1 \tag{19}$$

$$r'_{21}(\lambda, n) \to \frac{a_2}{a_1} \cdot \frac{n}{n+1},\ \lambda \to +\infty \tag{20}$$

(21) $$r'_{22}(\lambda, n) \to +\infty, \ \lambda \to +\infty.$$

For the equation ${}_{n+2}y''_2(x, a_0) = 0$ we obtain

$$r''_{22}(\lambda, n) = \frac{2a_2\lambda}{a_1} \cdot \frac{n}{n+2}\left(1 - \sqrt{1 - \frac{n^2+n-2}{n(n+1)} \cdot \frac{1}{\lambda}}\right), \ \lambda \geqslant 1$$

(22)

$$r'''_{22}(\lambda, n) = \frac{2a_2\lambda}{a_1} \frac{n}{n+2}\left(1 + \sqrt{1 - \frac{n^2+n-2}{n(n+1)} \cdot \frac{1}{\lambda}}\right), \ \lambda \geqslant 1$$

(23)

(24) $$r''_{21}(\lambda, n) \to \frac{a_2}{a_1} \cdot \frac{n-1}{n+1}, \ \lambda \to +\infty$$

(25) $$r''_{22}(\lambda, n) \to +\infty, \ \lambda \to +\infty.$$

The final conclusion on the distribution of the considered abscissas is as follows

$$0 < r''_{21}(\lambda, n) < r'_{21}(\lambda, n) < r_{21}(\lambda) < {}_2x_{\min}(\lambda) < r''_{22}(\lambda, n) < r'''_{22}(\lambda, n) < r_{22}(\lambda).$$

(26)

The inequality ${}_2x_{\min}(\lambda) < r''_{22}(\lambda, n)$ exists for $n \geqslant 3$ and for $\lambda > \frac{n^2 - n}{n^2 - n - 2}$.

All other inequalities are valid for $n = 1, 2, 3, \ldots$ and for each real number $\lambda > 1$. Inequalities (26) are proved by means of elementary calculations, whereby on starts from the formulae themselves for the mentioned abscissas.

Among the abscissas ${}_2x_{\min}(\lambda)$, $r''_{22}(\lambda, n)$, $r'_{22}(\lambda, n)$ and $r_{22}(\lambda)$ there exist the following asymptotic relations

(27) $$r_{22}(\lambda) - r'_{22}(\lambda, n) \sim \frac{4a_2\lambda}{(n+2)a_1}, \ \lambda \to +\infty$$

(28) $$r'_{22}(\lambda, n) - r''_{22}(\lambda, n) \sim \frac{4a_2\lambda}{(n+2)a_1}, \ \lambda \to +\infty$$

(29) $$r''_{22}(\lambda, n) - {}_2x_{\min}(\lambda) \sim \frac{2a_2\lambda}{a_1} \cdot \frac{n-2}{n+2}, \ \lambda \to +\infty, \ n \geqslant 3$$

(30) $$\frac{r_{22}(\lambda) - r'_{22}(\lambda, n)}{r'_{22}(\lambda, n) - r''_{22}(\lambda, n)} \sim 1, \quad \lambda \to +\infty,$$

hence there exists the

THEOREM (1). *The point $r'_{22}(\lambda, n)$ lies approximately in the middle between the points $r''_{22}(\lambda, n)$ and $r_{22}(\lambda)$ when $\lambda \to +\infty$.*

If in the formulae (14) and (18) we expand the square roots in a binomial series, we obtain

$$\frac{r_{22}(\lambda) - r'_{22}(\lambda, n)}{2\left(\frac{2a_2\lambda}{a_1}\right) \cdot \frac{1}{n+2}} = 1 + \frac{n+2}{2}\left\{-\binom{1/2}{1}\left(\frac{1}{\lambda}\right) + \binom{1/2}{2}\left(\frac{1}{\lambda}\right)^2 \mp \cdots\right\} -$$

$$- \frac{n+1}{2}\left\{-\binom{1/2}{1}\alpha + \binom{1/2}{2}\alpha^2 \mp \cdots\right\}, \ \alpha = \frac{n(n+2)}{(n+1)^2} \frac{1}{\lambda}, \ \lambda > 1,$$

from which relation (27). The argumentation for relation (28) is similar to that for relation (27).

2. *At the points $x = r''_{22}(\lambda, n)$ and $x = r'_{22}(\lambda, n)$ the ordinates of curve (12) increase indefinitely in absolute value when $\lambda \to +\infty$; for convenience of the calculation we can take $n = 2p - 1$ without diminishing the general case,*

$${}_{2p+1}y_2\{r'_{22}(\lambda, 2p-1), \lambda\} =$$

$$= -\lambda^{2p} \cdot \frac{4a_2}{a_1} \cdot \left\{\frac{(pa_1 + \sqrt{D})^{2p} \cdot (pa_1 + a_1 - \sqrt{D})}{(2p+1)^2} - \frac{a_2(pa_1 + \sqrt{D})^{2p-1}}{\lambda}\right\} \times$$

$$\times \left(\frac{4a_2}{(2p+1)a_1^2}\right)^{2p-1},$$

$$D = p^2 a_1^2 \left(1 - \frac{4p^2 - 1}{4p^2} \frac{1}{\lambda}\right), \text{ from which}$$

(31) $${}_{2p+1}y_2\{r'_{22}(\lambda, 2p-1), \lambda\} \to -\infty, \ \lambda \to +\infty.$$

The ordinate of the curve (8) at the point $x = r'_{22}(\lambda, n)$ increases indefinitely in absolute value when $\lambda \to +\infty$. This ordinate is

(32) $$y_{n+2}\{r'_{22}(\lambda, n)\lambda\} = {}_{n+2}y_2\{r'_{22}(\lambda, n), \lambda\}.$$

$$\cdot\left\{1 - \frac{a_3 - \frac{a_4}{r'_{22}(\lambda, n)} \pm \cdots - (-1)^{n+2} \cdot \frac{a_{n+2}}{(r'_{22}(\lambda, n))^{n-1}}}{r'_{22}(\lambda, n)[a_0 r'^2_{22}(\lambda, n) - a_1 r'_{22}(\lambda, n) + a_2]}\right\} \text{ from which}$$

(33) $$y_{n+2}\{r'_{22}(\lambda, n), \lambda\} \to -\infty, \quad \lambda \to +\infty.$$

In a quite similar manner we obtain

(34) $$y_{n+2}\{r''_{2}(\lambda, n), \lambda\} \to -\infty, \ \lambda \to +\infty;$$

moreover, we have also

(35) $$y_{n+2}\{{}_2x_{\min}(\lambda), \lambda\} = \left(\frac{2a_2\lambda}{a_1}\right)^n \cdot a_2(-\lambda + 1) \cdot$$

$$\cdot\left\{1 - \frac{a \cdot ({}_2x_{\min}(\lambda))^{n-1} \mp \cdots - (-1)^{n+2} a_{n+2}}{\left(\frac{2a_2\lambda}{a_1}\right)^n \cdot a_2(-\lambda + 1)}\right\} \to -\infty, \quad \lambda \to +\infty.$$

Finally, the ordinate of the curve (8) at the point $x = r_{22}(\lambda)$ increases indefinitely in absolute value when $\lambda \to +\infty$. The ordinate of the curve (8) at the point $x = r_{22}(\lambda)$ is

$$y_{n+2}\{r_{22}(\lambda), \lambda\} = r_{22}^{n}(\lambda)\{a_0 r_{22}^2(\lambda) - a_1 r_{22}(\lambda) + a_2\} - a_3 r_{22}^{n-1}(\lambda) +$$

$$\text{(36)} \qquad + a_4 r_{22}^{n-2}(\lambda) \mp \ldots + (-1)^{n+2} a_{n+2} = r_{22}^{n-1}(\lambda) \cdot \Big\{ -a_3 +$$

$$+ \frac{a_4}{r_{22}(\lambda)} \mp \ldots + (-1)^{n+2} \frac{a_{n+2}}{r_{22}^{n-1}(\lambda)} \Big\} \to -\infty, \quad \lambda \to +\infty.$$

3. *We shall represent certain theorems concerning the arc of the curve $y = y_{p+2}(x, \lambda)$ in the interval $(r''_{p+2,p}(\lambda), +\infty)$, where the point $x = r''_{p+2,p}(\lambda)$ is the last point of inflection.*

With regard to the asymptotic relation (10) we consider the sequence of parabolas

$$\text{(37)} \quad y = \eta_2(x, n) \equiv (n+2)\, a_0 x^2 - (n+1) a_1 x + n a_2, \quad n = 1, 2, 3, \ldots$$

All parabolas (37) pass through the fixed points $P(r_{21}, N_{21})$ and $Q(r_{22}, N_{22})$ whose ordinates are

$$\text{(38)} \quad N_{21}(a_0, n) \equiv \eta_2\{r_{21}(\lambda), n\} \equiv 2a_2(\lambda - \sqrt{\lambda^2 - \lambda - 1}) < 0, \ \lambda > 1,$$

$$\text{(39)} \quad N_{22}(a_0, n) \equiv \eta_2\{r_{22}(\lambda), n\} = 2a_2(\lambda + \sqrt{\lambda^2 - \lambda - 1}) > 0, \ \lambda > 1;$$

it is easy to prove that these quantities are not dependent on n and therefore we write them without the argument n, for instance

$$\text{(40)} \quad N_{22}(a_0) \equiv N_{22}(a_0, n) = N_{22}(a_0, 1) = 3a_0 r_{22}^2(\lambda) - 2a_1 r_{22}(\lambda) + a_2.$$

If we use the parameter λ, then we write $N_{22}(\lambda)$. For $\lambda = 1$ we have $N_{22}(1) = 0$. In virtue of (15) and (17) we have

$$\text{(41)} \quad \lim_{\lambda \to +\infty} N_{22}(\lambda) = +\infty.$$

With regard to the asymptotic relation (11) we consider the sequence of parabolas

$$y = \zeta_2\{(x, n) \equiv \binom{n+2}{2} a_0 x^2 - \binom{n+1}{2} a_1 x + \binom{n}{2} a_2, \quad n = 1, 2, 3 \ldots$$

(42)

In virtue of the formula $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ we have

$$\zeta_2(x, n+1) = \zeta_2(x, n) + \eta_2(x, n)$$

and from the last formula it follows immediately

$$\text{(43)} \quad \zeta_2(x, n+1) = \zeta_2(x, 1) + \sum_{\nu=0}^{n} \eta_2(x, \nu)$$

and from this $\zeta_2\{r_{22}(\lambda), n+1\} = \zeta_2\{r_{22}(\lambda), 1\} + n \cdot N_{22}(\lambda) =$

$$\text{(44)} \quad a_2\{2(n+2)(\lambda + \sqrt{\lambda^2 - \lambda - 1}) + 1\} \to +\infty, \ \lambda \to +\infty.$$

The tangent to the curve $y = y_{p+2}(x, \lambda)$ at the point $x = r_{22}(\lambda)$ forms an acute angle which tends to $\frac{\pi}{2}$ as $\lambda \to +\infty$. The first derivate of this function at the point $x = r_{22}(\lambda)$ is

$$\text{(45)} \quad y'_{p+2}\{r_{22}(\lambda), \lambda\} =$$

$$r_{22}^{p-1}(\lambda) \cdot N_{22}(\lambda) \cdot \Big\{1 - \frac{(p-1)\, a_3 r_{22}^{p-2}(\lambda) \mp \ldots - (-1)^{p+1} a_{p+1}}{r_{22}^{p-1}(\lambda) \cdot N_{22}(\lambda)}\Big\} \to +\infty,$$

$$\lambda \to +\infty,$$

from which it follows

$$\text{(46)} \quad y'_{p+2}\{r_{22}(\lambda_0), \lambda_0\} = A(\lambda_0) > 0,$$

where $A(\lambda_0)$ is a positive and arbitrarily great number depending on λ_0, while λ_0 is a sufficiently great value of the parameter λ.

Since $y'_{p+2}(x, \lambda_0)$ is a polynomial, we have, by a known theorem,

$$y'_{p+2}(x, \lambda_0) \to +\infty, \ x \to +\infty$$

and in virtue of this there exists in the neighbourhood of the pont $x = +\infty$ such a x' for which we have

$$\text{(47)} \quad 1. \ y'_{p+2}(x', \lambda_0) > A(\lambda_0)$$

$$\text{(48)} \quad 2. \ x' = r_{22}(\lambda') > 0, \quad \lambda' > \lambda_0.$$

In virtue of (13) we have

$$\text{(49)} \quad 0 < r_{22}(\lambda_0) < r_{22}(\lambda') \text{ for } \lambda' > \lambda_0,$$

from which, in virtue of (48), it follows

$$\text{(50)} \quad 0 < r_{22}(\lambda_0) < x'.$$

From (47) and (50) it follows

THEOREM (2). *For all values $\lambda' \geqslant \lambda_0$, where λ_0 is a sufficiently great value of the parameter λ, the parabola $y = y_{p+2}(x, \lambda')$ increases monotonously to the right of the straight line $x = r_{22}(\lambda')$.*

From (36) it follows

$$\text{(51)} \quad y_{p+2}\{r_{22}(\lambda_0), \lambda_0\} = -R(\lambda_0) < 0.$$

The value λ_0 can so be chosen as to be the same in relations (46) and (51). By theorem (2) the parabola $y = y_{p+2}(x, \lambda)$ is monotonously increasing to the right of the straight line $x = r_{22}(\lambda)$ and the ordinate (51) is negative, thus the point of intersection $r_{p+2,p+2}(\lambda)$ of this parabola with the axis X_{p+2} is situated to the right of the point $r_{22}(\lambda)$, namely

(52) $$0 < r_{22}(\lambda) < r_{p+2,p+2}(\lambda).$$

Owing to (16), from (52) it follows

(53) $$r_{p+2,p+2}(\lambda) \to +\infty, \quad \lambda \to +\infty.$$

The first derivate of the parabola $y = y_{p+2}(x, \lambda)$ at the point $x = r'_{22}(\lambda, p)$ is

$$y'_{p+2}\{r'_{22}(\lambda, p), \lambda\} = \{r'_{22}(\lambda, p)\}^{p-2} \cdot \left\{-(p-1)a_3 + \frac{(p-2)a_4}{r'_{22}(\lambda, p)} \mp \dots + (-1)^{p+1} \cdot \frac{a_{p+1}}{(r'_{22}(\lambda, p))^{p-2}}\right\},$$

from which it follows

$$y'_{p+2}\{r'_{22}(\lambda, p), \lambda\} \to -\infty, \quad \lambda \to +\infty,$$

thus

(54) $$y'_{p+2}\{r'_{22}(\lambda_0, p), \lambda_0\} = -B(\lambda_0) < 0.$$

The value λ_0 can so be chosen as to be the same in relations (46), (52) and (54). From (46) and (54) it follows that in the interval $\{r'_{22}(\lambda_0, p), r_{22}(\lambda_0)\}$ the first derivate $y'_{2p+2}(x, \lambda_0)$ changes the sign and takes the value 0, thus

(55) $$y'_{p+2}\{r'_{p+2,p+1}(\lambda_0), \lambda_0\} \equiv 0,$$

whereby $r'_{p+2,p+1}(\lambda_0)$ is the greatest root of the equation $y'_{p+2}(x, \lambda_0) = 0$. Owing to (54) and (46), for the abscissa $r'_{p+2,p+1}(\lambda_0)$ there holds the inequality

(56) $$0 < r'_{22}(\lambda_0, p) < r'_{p+2,p+1}(\lambda_0) < r_{22}(\lambda_0).$$

The second derivate of the parabola $y = y_{p+2}(x, \lambda)$ at the point $x = r''_{22}(\lambda, p)$ is

$$y''_{p+2}\{r''_{22}(\lambda, p), \lambda\} = -2\binom{p-1}{2}a_3 \cdot \{r''_{22}(\lambda, p)\}^{p-3} \cdot \left\{1 - \frac{\binom{p-2}{2}a_4}{\binom{p-1}{2}a_3} \cdot \frac{1}{r''_{22}(\lambda, p)} \pm \dots + (-1)^p \cdot \frac{\binom{2}{2}a_p}{\binom{p-1}{2}a_3} \cdot \frac{1}{(r''_{22}(\lambda, p))^{p-3}}\right\};$$

(57) $$y''_{p+2}\{r''_{22}(\lambda, p), \lambda\} \to -\infty, \quad \lambda \to +\infty, \text{ from which}$$

$$y''_{p+2}\{r''_{22}(\lambda_0, p), \lambda_0\} = -C(\lambda_0) < 0.$$

Consequently the parabola $y = y_{p+2}(x, \lambda_0)$ is convex at the point $x = r''_{22}(\lambda_0, p)$ for a sufficiently great value λ_0.

The second derivate of the parabola $y = y_{p+2}(x, \lambda)$ at the point $x = r'_{22}(\lambda, p)$ is

(58) $$y''_{p+2}\{r'_{22}(\lambda, p), \lambda\} = 2\{r'_{22}(\lambda, p)\}^{p-2} \cdot \zeta_2\{r'_{22}(\lambda, p), p\} \cdot \left\{1 - \frac{\binom{p-2}{2} a_3 r'^{\,p-3}_{22}(\lambda, p) \mp \dots - (-1)^p \binom{2}{2} a_p}{r'^{\,p-2}_{22}(\lambda, p) \cdot \zeta_2\{r'_{22}(\lambda, p), p\}}\right\},$$

$$\zeta_2\{r'_{22}(\lambda, p), p\} = a_2\left\{\frac{(p+1)^2}{p+2}\left(\lambda + \sqrt{\lambda^2 - \frac{p(p+2)}{(p+1)^2} \cdot \frac{1}{\lambda}}\right) - p\right\} \to +\infty, \quad \lambda \to +\infty;$$

$$y''_{p+2}\{r'_{22}(\lambda, p), \lambda\} \to +\infty, \quad \lambda \to +\infty, \text{ from which}$$

(59) $$y''_{p+2}\{r'_{22}(\lambda_0, p), \lambda_0\} = +K(\lambda_0) > 0.$$

Consequently the parabola $y = y_{p+2}(x, \lambda_0)$ is concave at the point $x = r'_{22}(\lambda_0, p)$.

From (57) and (59) it follows that in the interval $\{r''_{22}(\lambda_0, p), r'_{22}(\lambda_0, p)\}$ the second derivate changes the sign and takes the value 0, thus

(60) $$y''_{p+2}\{r''_{p+2,p}(\lambda_0), \lambda_0\} \equiv 0,$$

whereby $r''_{p+2,p}(\lambda_0)$ is the greatest root of the equation $y''_{p+2}(x, \lambda_0) = 0$. Owing to (57) and (59), for the abscissa $r''_{p+2,p}(\lambda_0)$ there holds the inequality

(61) $$0 < r''_{22}(\lambda_0, p) < r''_{p+2,p}(\lambda_0) < r'_{22}(\lambda_0, p).$$

Since $r''_{p+2,p}(\lambda)$ is the last real zero of the polynomial $y''_{p+2}(x, \lambda)$, this means that $y''_{p+2}(x, \lambda)$, taking only positive values, tends to $+\infty$ as x increases from $r''_{p+2,p}(\lambda)$ to $+\infty$. Hence the second derivate $y''_{p+2}(x, \lambda)$ remains permanently positive as x increases from $r''_{p+2,p}(\lambda)$ to $+\infty$, and this means that the parabola is concave in the interval $\{r''_{p+2,p}(\lambda), +\infty\}$. Consequently in the interval $\{r''_{p+2,p}(\lambda), +\infty\}$ there are no waves on the arc of the parabola $y = y_{p+2}(x, \lambda)$ and therefore, when λ increases, no new zero can arise behind the zero $r_{p+2,p+2}(\lambda)$. This is the concept of the last zero. Thus we have the

THEOREM (3). *For a sufficiently large $\lambda' \geqslant \lambda_0$ the curve $y = y_{p+2}(x, \lambda')$ cuts the axis X_{p+2} at the point $x = r_{p+2,p+2}(\lambda')$ and when λ' continues to increase, no new zero can arise behind this zero.*

At the point $D\{r_{22}(\lambda), y_{p+2}[r_{22}(\lambda), \lambda]\}$ of the system (X_{p+2}) the tangent to the curve $y = y_{p+2}(x, \lambda)$ is, by (46), arbitrarily steep; since this curve is concave in the interval $\{r''_{p+2,p}(\lambda), +\infty\}$, it follows that this curve lies above the tangent at the point D. If we denote by $d_{p+2}(\lambda)$ the abscissa of the point of intersection of this tangent with the axis X_{p+2}, we have

(62) $$0 < r_{22}(\lambda) < r_{p+2,p+2}(\lambda) < d_{p+2}(\lambda), \quad \lambda > \lambda_0.$$

The point of intersection if the mentioned tangent with the axis X_{p+2} has the abscissa

(63) $$x = d_{p+2}(\lambda) = r_{22}(\lambda) - \frac{y_{p+2}\{r_{22}(\lambda), \lambda\}}{y'_{p+2}\{r_{22}(\lambda), \lambda\}}$$

In virtue of (16) and (45) we have

$$\frac{y_{p+2}\{r_{22}(\lambda), \lambda\}}{y'_{p+2}\{r_{22}(\lambda), \lambda\}} \to 0, \ \lambda \to +\infty$$

and therefore from (63) it follows that

(64) $$\{d_{p+2}(\lambda) - r_{22}(\lambda)\} \to 0 \text{ as } \lambda \to +\infty.$$

From (62), in virtue of (64), we obtain

(65) $$r_{22}(\lambda) \sim r_{p+2,p+2}(\lambda) \text{ as } \lambda \to +\infty.$$

The equations $y'_{p+2}(x, \lambda) = 0$ and $(p+2)a_0x^2 - (p+1)a_1x + a_2 = 0$ belong to the same finite sequence of equations (5) since their first three terms are equal. Therefore to their greatest roots there can be applied the asymptotic relation (65) which now has the following fomr

(66) $$0 < r'_{p+2,p+1}(\lambda) \sim r'_{22}(\lambda, p), \ \lambda \to +\infty.$$

In the same manner the equations

$$y''_{p+2}(x, \lambda) = 0 \text{ and } \binom{p+2}{2} a_0x^2 - \binom{p+1}{2} a_1x + \binom{p}{2} a_2 = 0$$

give the relation

(67) $$0 < r''_{p+2,p}(\lambda) \sim r''_{22}(\lambda, p), \ \lambda \to +\infty.$$

By (33), (34), (35) and (36) the absolute values of the ordinates of the parabola $y = y_{p+2}(x, \lambda)$ at the points $x = {}_2x_{\min}(\lambda), < r''_{22}(\lambda, p) < < r'_{22}(\lambda, p) < r_{22}(\lambda)$ increase indefinitely with λ and according to the aforesaid the arc of this parabola is concave in the iterval $\{r''_{p+2,p}(\lambda), r_{p+2,p+2}(\lambda)\}$.

If the square roots in formulae (14) and (23) are expanded in series or formulae (27) and (28) are added up, we obtain

$$r_{22}(\lambda) - r''_{22}(\lambda, n) \sim \frac{8a_2\lambda}{(n+2)a_1}, \ \lambda \to +\infty$$

and in view of the asymptotic relations (65) and (67) we have

(68) $$\{r_{p+2,p+2}(\lambda) - r''_{p+2,p}(\lambda)\} \to +\infty, \ \lambda \to +\infty.$$

The arc of the parabola $y = y_{p+2}(x, \lambda)$ in the interval $\{r''_{p+2,p}(\lambda), r_{p+2,p+2}(\lambda)\}$ is called the „last arc" of this parabola. Thus, the following theorem has been proved:

THEOREM (4): *The „last arc" of the parabola $y = y_{p+2}(x, \lambda)$ lies below the axis X_{p+2} and the absolute value of each ordinate of this arc as well as the length $\{r_{p+2,p+2}(\lambda) - r''_{p+2,p}(\lambda)\}$ of the interval $\{r''_{p+2,p}(\lambda), r_{p+2,p+2}(\lambda)\}$ increase indefinitely with λ.*

In view of relations (65), (66) and (67), from the asymptotic relation (30) we obtain

(69) $$\frac{r_{p+2,p+2}(\lambda) - r'_{p+2,p+1}(\lambda)}{r'_{p+2,p+1}(\lambda) - r''_{p+2,p}(\lambda)} \sim 1, \ \lambda \to +\infty.$$

4. The inequality $r_{2p,2p}(\lambda) < r_{2p+1,2p+1}(\lambda)$ and the asymptotic relation $r_{2p,2p}(\lambda) \sim r_{2p+1,2p+1}(\lambda)$.

In virtue of teorem (4) it follows: The „last arc" of the parabola $y = y_{2p}(x, \lambda)$ touches the axis X_{2p} from the side of the positive ordinates for the value $\lambda = \alpha_{2p}$ of the parameter λ. From this we deduce the

THEOREM (5). *If the coefficient a_0 of a polynomial of even degree decreases and if all the zeros of this polynomial are complex, then this polynomial has at least one double real zero for an exactly determined value $\lambda = \alpha_{2p}$ of the parameter λ and to the right of this zero no new zero can arise when λ continues to increase indefinitely.*

It is known from the theory of algebraic equations [5]: the presence of a positive minimum or of a negative maximum of a polynomial means the existence of two conjugate complex roots. Consequently two conjugate complex roots are bound to the „last arc" if this arc does not cut the axis X_{2p}. Hence we have

THEOREM (6). *The two conjugate complex roots which are bound to the positive minimum of the „last arc" of the parabola $y = y_{2p}(x, \lambda)$ vanish if the parameter λ takes the value $\lambda = \alpha_{2p}$ and these two conjugate complex roots change to one double real root.*

As long as the parameter λ has a value which is infinitely little different from and smaller than α_{2p}, the curve $y = y_{2p}(x, \lambda)$, $\lambda = \alpha_{2p} - \varepsilon$, $(\varepsilon \to 0)$ is arbitrarily close to and above the axis X_{2p} in the neighbourhood of the last double real zero. Thus, for $\lambda = \alpha_{2p} - \varepsilon$ the complex zeros belonging to the positive minimum are situated in the neighbourhood of the last double real zero, while the other complex zeros, if they exist, have smaller real parts.

For $\lambda > \alpha_{2p}$ the „last arc" of the curve

(70) $$y = y_{2p+1}(x, \lambda) = x \cdot y_{2p}(x, \lambda) - a_{2p+1}, \ \lambda > \alpha_{2p},$$

passes through the points $r_{2p,2p-1}(\lambda)$ and $r_{2p,2p}(\lambda)$ of the axis X_{2p} and lies below this axis in the interval between these two points. The arc of the curve (70) is in the interval $\{r_{2p,2p}(\lambda), +\infty\}$ concave and monotonously increasing. Owing to this, the curve (70) cuts the axis X_{2p+1}, lying above the axis X_{2p}, at the point $r_{2p+1,2p+1}(\lambda)$ situated to the right of the straight line $x = r_{2p,2p}(\lambda)$, $\lambda > \alpha_{2p}$, i.e.

(71) $$0 < r_{2p,2p}(\lambda) < r_{2p+1,2p+1}(\lambda), \quad \lambda > \alpha_{2p}.$$

The point of intersection of the tangent to the curve $y = y_{2p+1}(x, \lambda)$, at the point $T\{r_{2p,2p}(\lambda), -a_{2p+1}\}$ of the system (X_{2p+1}), with the axis X_{2p+1} has the abscissa

(72) $$t_{2p+1}(\lambda) = r_{2p,2p}(\lambda) - \frac{a_{2p+1}}{y'_{2p+1}\{r_{2p,2p}(\lambda), \lambda\}}, \ \lambda > \alpha_{2p}.$$

We wish to prove that $y'_{2p+1}\{r_{2p,2p}(\lambda), \lambda\}$ increases indefinitely with λ. By (53) we have

(73) $$r_{2p,2p}(\lambda) \to +\infty, \quad \lambda \to +\infty$$

and by (52)

$$0 < r_{22}(\lambda) < r_{2p,2p}(\lambda), \quad \lambda > \alpha_{2p}. \tag{74}$$

The value of the first derivate $y'_{2p+1}(x, \lambda)$ at the point $x = r_{2p,2p}(\lambda)$ is

$$y'_{2p+1}\{r_{2p,2p}(\lambda), \lambda\} = r_{2p,2p}^{2p-2}(\lambda) \cdot \eta_2\{r_{2p,2p}(\lambda), 2p-1\} \cdot \left\{1 - \frac{(2p-2)a_3 r_{2p,2p}^{2p-3}(\lambda) \mp \ldots - (-1)^{2p} a_{2p}}{r_{2p,2p}^{2p-2}(\lambda) \cdot \eta_2\{r_{2p,2p}(\lambda), 2p-1\}}\right\}. \tag{75}$$

The parabola

$$y = \eta_2(x, 2p-1) \equiv (2p+1)a_0 x^2 - 2pa_1 x + (2p-1)a_2 \tag{76}$$

cuts the axis X_2 at the points $x_1 = r'_{21}(\lambda, 2p-1)$ and $x_2 = r'_{22}(\lambda, 2p-1)$. In virtue of the inequality (26) we have

$$0 < r'_{22}(\lambda, 2p-1) < r_{22}(\lambda), \quad \lambda > 1,$$

and with regard to the inequality (74) it follows

$$0 < r'_{22}(\lambda, 2p-1) < r_{22}(\lambda) < r_{2p,2p}(\lambda), \quad \lambda > \alpha_{2p}. \tag{77}$$

Parabola (76), for $x = r_{22}(\lambda)$, has the ordinate

$$\eta_2\{r_{22}(\lambda), 2p-1\} = 2a_2(\lambda + \sqrt{\lambda^2 - \lambda} - 1) \to +\infty, \quad \lambda \to +\infty. \tag{78}$$

The ordinates of parabola (76) are positive for $x > r'_{22}(\lambda, 2p-1)$ and they increase monotonously as $x \to +\infty$. Thus, owing to (74), there exists the inequality

$$\eta_2\{r_{2p,2p}(\lambda), 2p-1\} > \eta_2\{r_{22}(\lambda), 2p-1\} > 0, \quad \lambda > \alpha_{2p}$$

and, owing to (78), from this it follows

$$\eta_2\{r_{2p,2p}(\lambda), 2p-1\} + \infty, \quad \lambda \to +\infty. \tag{79}$$

From (75) and (79) it follows

$$y'_{2p+1}\{r_{2p,2p}(\lambda), \lambda\} \to +\infty, \quad \lambda \to +\infty. \tag{80}$$

The formulae (72) and (80) give

$$\{t_{2p+1}(\lambda) - r_{2p,2p}(\lambda)\} \to 0, \quad \lambda \to +\infty \tag{81}$$

and, since the curve $y = y_{2p+1}(x, \lambda)$ is concave in the interval $\{r''_{2p+1,2p-1}(\lambda), +\infty\}$, in virtue of this concavity and in virtue of (71) we obtain

$$0 < r_{2p,2p}(\lambda) < r_{2p+1,2p+1}(\lambda) < t_{2p+1}(\lambda), \quad \lambda > \alpha_{2p}. \tag{82}$$

Thus, from (82), in virtue of (81), we obtain

$$r_{2p,2p}(\lambda) \sim r_{2p+1,2p+1}(\lambda), \quad \lambda \to +\infty. \tag{83}$$

For the negative roots of the greatest absolute value $r_{2n+1,1}(\lambda)$ and $r_{2n,1}(\lambda)$ of real equations with positive coefficients

$$\sum_{\nu=0}^{p} a_\nu x^{p-\nu} = 0, \quad p = 2n, 2n+1$$

it can be proved by means of the same method [4] that for them there exists the asymptotic relation

$$r_{2n+1,1}(\lambda) \sim r_{2n,1}(\lambda), \quad \lambda \to +\infty.$$

Since the proofs for the inequalities $r_{2p,2p}(\lambda) < r_{2p+2,2p+2}(\lambda)$ and $r_{2p+3,2p+3}(\lambda) < r_{2p+1,2p+1}(\lambda)$ are very lengthy, we shall give them in separate paper. It should be mentioned that the proofs for these inequalities have their basis in Theorem (4).

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