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# AN APPROXIMATION OF THE HELMHOLTZ EQUATION ON FINITE ELEMENTS AND DIFFERENCES

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1. **Introduction.** a) *The Helmholtz Equation.* Let us consider the Helmholtz equation with the Dirichlet condition:

$$\text{(1)}\quad Au \equiv -\frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q\frac{\partial u}{\partial y}\right) + ru(x, y) = f(x, y), \quad (x, y) \in \Omega$$

$$\text{(1a)}\quad u = g(x, y),\ (x, y) \in \Gamma$$

where $u$ is the unknown function inside the convex domain $\Omega$ of $R^2$ with the boundary $\Gamma$ (sufficiently smooth) and $p, q, r, f$ and $g$ are prescribed bounded functions with properties to provide a unique solution for $u$ in a space in which the problem (1)–(1$a$) is to be solved. $D(A)$ will be the definition domain of the Helmholtz operator $A$.

*Remarks.* It may presumed that $\partial p/\partial x$, $\partial q/\partial y$, $r$, $f$, $g \in C(\bar{\Omega})$ and $r(x, y) \geqslant 0$, $p(x, y) > 0$, $q(x, y) > 0$, $(x, y) \in \Omega$; then, there exists the solution $u \in D(A) = \{u | u \in C^2(\Omega),\ u_\Gamma = g\}$. In the same way, for example, for the solution $u \in H^2(\Omega)$ the conditions $f \in L_2(\Omega)$ and $g \in H^{3/2}(\Omega)$ – with $p = q = r = \text{const} > 0$ – are necessary and sufficient.

– We consider $D(A)$ to be within the linear space of functions

$$H^1(\Omega) = \left\{w \,|\, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y} \in L_2(\Omega)\right\}$$

in which the scalar product $(.,.)$, the norm $\|\cdot\|$ as well as the symmetrical bilinear form $a(u, v)$ are defined by the equalities

$$(u, v)_{H^1} = \iint_\Omega (uv + \nabla u \cdot \nabla v)\, dx\, dy, \quad \|u\|^2 = \iint_\Omega (u^2 + |\nabla u|^2)\, dx\, dy$$

$$a(u, v) = \iint_\Omega \left(p\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + q\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + ruv\right) dx\, dy$$

The following liear spaces are also introduced

$$H_0^1(\Omega) = \{w | w \in H^1(\Omega),\ w_\Gamma = 0\}, \quad H_g^1 = \{w | w \in H^1(\Omega),\ w_\Gamma = g,\ \ g \in H^{1/2}(\Gamma)$$

*Remark.* $H^1(\Omega)$ is a Hilbert space (the energy space of the operator $A$, where $D(A)$ is dense).

We shall consider the problem in weak (generalized) formulation in the space $H^1_g(\Omega)$ (and not in $D(A)$).

PROPOSITION 1. *The Helmholtz equation is written in the integral form (weak formulation)*

$$a(u, v) = (f, v), \forall v \in H^1_0(\Omega), \; u \in H^1_g(\Omega) \tag{2}$$

*Proof.* We perform a scalar multiplication of the $L_2$ type of equation (1) by an arbitrary function $v \in H^1_0(\Omega)$; the Green formula and the condition $v_\Gamma = 0$ are applied, subsequently.

DEFINITION. *The unique solution* $u \in H^1_g(\Omega)$ *of equation* (2) *is called a generalized solution of the problem* (1) — (1*a*).

PROPOSITION 2. *The generalized solution of the problem* (1)—(1*a*) *if* $p$, $q$, $r$ *are non-negative minimizes the energy functional*

$$F(u) = a(u, u) - 2(f, u), \; u \in H^1_g(\Omega) \tag{3}$$

*Proof.* Let us consider $u \in H^1_g(\Omega)$ the generalized solution and the function $w = u + \alpha v$ with $v \in H^1_0(\Omega)$ and $\alpha \in R^1$ an arbitrary parameter (thus, $w \in H^1_g(\Omega)$). We have

$$F(w) = F(u + \alpha v) = a(u, u) + \alpha^2 a(v, v) + 2\alpha\, a(u, v) - 2(f, u) - 2\alpha (f, v). \tag{4}$$

Hence, since we have $a(u, v) = (f, v), \forall v \in H^1_0(\Omega)$, we get

$$F(w) - F(u) = \alpha^2 a(v, v) \geqslant 0 \text{ if } p, q, r \geqslant 0$$

Now, let us consider (reciprocally) $u \in H^1_g(\Omega)$ the function that minimizes the energy functional $F$, (3), and $w = u + \alpha v$, $\alpha \in R^1$ admissible functions [from $H^1_g(\Omega)$; thus $v \in H^1_0(\Omega)$]. The functional $F(u + \alpha v)$ for fixed $u$ and $v$ ($u$ — extremal and $v$ arbitrary fixed) becomes a function of $\alpha$ which is to be minimal for $\alpha = 0$. Therefore, we necessarily have $\partial F/\partial \alpha = 0$ for $\alpha = 0$ which leads to (see (4)) the equality $a(u, v) - (f, v) = 0, \forall v \in H^1_0(\Omega)$ which proves that the extremal $u$ of $F$ is a generalized solution of the problem (1)—(1a).

**2. The Approximation of the Helmholtz Equation.** a) *The Ritz variational procedure on rectangular finite elements.* The $\Omega$ domain is discretized by the straight lines $x = x_i = ih$, $y = y_j = jk$ into rectangular finite elements whose vertices form the node grid $\overline{\Omega}_h = \{(x_i, y_j), i = \overline{0, I}, j = \overline{0, J}; h, k = \text{constant}\}$. We consider a rectangular element $(e_1)$ and a local cartesian frame $C\xi\eta$ (fig. 1), with the axis $C\xi$ and $C\eta$ parallel to $Ox$ and respectively $Oy$ and the origin in the centre $C(x_c, y_c)$ of the element. For the point $P(x, y) \in (e_1)$ we have $x = x_c + \xi$, $y = y_c + \eta$. We introduce the local dimensionless coordinates in $(e_1)$

$$\xi = 2\frac{x - x_c}{h}, \quad \eta = 2\frac{y - y_c}{k}$$



Fig. 1 shows the coordinates of the vertices of $(e_1)$ in $C\xi\eta$. Let us approximate the exact solution $u \in H^1_g(\Omega)$($\Omega$ is bounded by a rectangle) of the variational problem for $F(u)$, with functions from the finite dimensional space (from $H^1(\Omega)$):

$$H^{1,h} = \{U \mid U \in H^1(\Omega),\; U = \alpha_0^{(i)} + \alpha_1^{(i)}x + \alpha_2^{(i)}y + \alpha_3^{(i)}xy \text{ on } (e_i),\; i = \overline{1, N}\},$$

where $N$ is the number of the elements in $\Omega$ and $\alpha_g^{(i)} \in R^1$

Therefore, on $(e_1)$ in the local system $C\xi\eta$, $u$ will be approximated through the bilinear interpolation polynomial (for $U$)

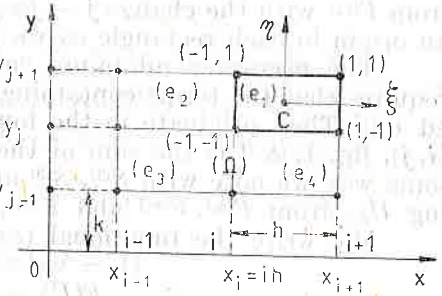


Fig. 1

$$\begin{aligned} U(\xi, \eta) = \frac{1}{4} [&(1 - \xi)(1 - \eta)u_{i,j} + (1 + \xi)(1 - \eta)u_{i+1,j} + (1 + \\ &+ \xi)(1 + \eta)u_{i+1,j+1} + \\ &+ (1 - \xi)(1 + \eta)u_{i,j+1}]; \end{aligned} \tag{5}$$

$$(i = \overline{1, I-1}, j = \overline{1, J-1})$$

where, as $u$ and $U$ are continuous functions on $\Omega$, $u_{i,j} = U(x_i, y_j) = u(x_i, y_j)$; $u_{i,j}$ are finite values on $\Omega_h$ for all $i$ and $j$.

On $(e_1)$ the minimization functional $F$, for the function $U$ has the value

$$F^{(e_1)}(U) = \iint\limits_{(e_1)} \left[ p\left(\frac{\partial U}{\partial x}\right)^2 + q\left(\frac{\partial U}{\partial y}\right)^2 + rU^2 - 2fU \right] dx\, dy \tag{6}$$

where ($F^{(e_1)}(U) = T_1 + T_2 + T_3 + T_4$; $(\partial U/\partial x)^2 \equiv U_x^2$ etc):

$$\begin{aligned} T_1 &= \iint\limits_{(e_1)} pU_x^2 dx\, dy = \frac{k}{3h} [(u_{i+1,j} - u_{i,j})^2 + (u_{i+1,j+1} - u_{i,j+1})^2 + \\ &\quad + (u_{i+1,j} - u_{i,j})(u_{i+1,j+1}) - u_{i,j+1})] p_{i+1/2,j+1/2}; \\ T_2 &= \iint\limits_{(e_1)} qU_y^2 dx\, dy = \frac{h}{3k} [(u_{i,j+1} - u_{i,j})^2 + (u_{i+1,j+1} - u_{i+1,j})^2 + \\ &\quad + (u_{i+1,j+1} - u_{i+1,j})(u_{i,j+1} - u_{i,j})] q_{i+1/2,j+1/2}; \\ T_3 &= -\iint\limits_{(e_1)} fu\, dx\, dy = -\frac{hk}{2} (u_{i,j} + u_{i+1,j} + u_{i+1,j+1} + u_{i,j+1}) f_{i+1/2,j+1/2}; \\ T_4 &= \iint\limits_{(e_1)} rU^2\, dx\, dy = \frac{hk}{18} [2(u_{i,j}^2 + u_{i+1,j}^2 + u_{i+1,j+1}^2 + u_{i,j+1}^2) + \\ &\quad + 2u_{i+1,j}u_{i,j} + u_{i,j}u_{i+1,j+1} + 2u_{i,j}u_{i,j+1} + 2u_{i+1,j}u_{i+1,j+1} + \\ &\quad + u_{i+1,j}u_{i,j+1} + 2u_{i+1,j+1}u_{i,j+1}] r_{i+1/2,j+1/2} \end{aligned} \tag{7}$$

On the whole domain the minimal value $F(U)$ of the functional will be

$$(8)\qquad F(U) = \sum_{i=1}^{N} F^{(e_i)}(U)$$

where $N$ is the number of the rectangular elements in $\Omega$. The value $F^{(e_2)}$ is obtained form $F^{(e_1)}$ by the index substitution $i \to i-1$; $F^{(e_3)}$ is obtained from $F^{(e_1)}$ with the change $i \to i-1$, $j \to j-1$; $F^{(e_4)}$ is obtained from $F^{(e_1)}$ with the change $j \to j-1$ (the bottom left corner is chosen as an origin in each rectangle as we have done in $(e_1)$.

The necessary minimum condition (stationarity) $\partial F(U)/\partial U_{i,j} = 0$ requires that the terms containing the nodal value $U_{i,j}$ should be pointed out. They originate in the four rectangles having a common vertex $(i, j)$, fig. 1. $S^{(1)}$ is the sum of the terms in $F^{(e_1)}$ which contain $U_{i,j}$. In some way we note with $S^{(2)}$, $S^{(3)}$ and $S^{(4)}$ the sum of the terms containing $U_{i,j}$ from $F^{(e_2)}$, $F^{(e_3)}$ and $F^{(e_4)}$.

We write the functional (8) in the form

$$F(U) = \sum_{m=1}^{4} S^{(m)} + \Sigma_c$$

where $\Sigma_c$ stands for the sum of the terms which do not contain $U_{i,j}$.

The necessary minimum condition is written in the form

$$(10)\qquad \sum_{m=1}^{4} \frac{\partial S^m}{\partial U_{i,j}} = 0,\ (i = \overline{1, I}\,;\ j = \overline{1, J})$$

We calculate the derivatives in (10) by using $F^{(em)}$, with $m = \overline{1, 4}$, in which only $S^{(m)}$ is taken into consideration:

$$\frac{\partial S^{(1)}}{\partial U_{i,j}} = \frac{\partial F^{(e_1)}}{\partial U_{i,j}} = \frac{k}{3h}[-2(u_{i+1,j} - u_{i,j}) - u_{i+1,j+1} + u_{i,j+1}] \times$$
$$\times\, p_{i+1/2,j+1/2} + \frac{h}{3k}[-2(u_{i,j+1} - u_{i,j}) - u_{i+1,j+1} + u_{i+1,j}]q_{i+1/2,j+1/2} -$$
$$- \frac{hk}{2} f_{i+1/2,j+1/2} + \frac{hk}{18}[4u_{i,j} + 2u_{i+1,j} + u_{i+1,j+1} + 2u_{i,j+1}]r_{i+1/2,j+1/2};$$

$$\frac{\partial S^{(2)}}{\partial U_{i,j}} = \frac{k}{3h}[2(u_{i,j} - u_{i-1,j}) + u_{i,j+1} - u_{i-1,j+1}]\,p_{i-1/2,j+1/2} +$$
$$+ \frac{h}{3k}[-2(u_{i,j+1} - u_{i,j}) - u_{i-1,j+1} + u_{i-1,j}]q_{i-1/2,j+1/2} -$$
$$- \frac{hk}{2}[f_{i-1/2,j+1/2} - \frac{1}{9}(4u_{i,j} + 2u_{i-1,j} + 2u_{i,j+1} + u_{i-1,j+1})r_{i-1/2,j+1/2}];$$

$$\frac{\partial S^{(3)}}{\partial U_{i,j}} = \frac{k}{3h}[2(u_{i,j} - u_{i-1,j}) + u_{i,j-1} - u_{i-1,j-1}]\,p_{i-1/2,j-1/2} +$$
$$+ \cdots\cdots\cdots\cdots;$$

$$\frac{\partial S^{(4)}}{\partial U_{i,j}} = \frac{k}{3h}[2(u_{i,j} - u_{i+1,j}) - u_{i+1,j-1}u_{i,j-1}]\,p_{i+1/2,j-1/2} +$$
$$+ \cdots\cdots\cdots\cdots$$



By using these derivatives, condition (10) turns into an equation that has nine unknown nodal values, the value $u_{i,j}$ as well as eight values from the nodes neighbouring the node $(i, j)$. This equation may be interpreted as a finite difference equation.

The Helmholtz equation may be approximated at the point $(i, j)$ by the nine-node finite-difference equation:

$$\Delta_9\{u_{i,j}\} \equiv$$
$$(12)\qquad a_{i-1/2,j+1/2}u_{i-1,j+1} + b_{i-1/2,j+1/2}^{i+1/2,j+1/2}u_{i,j+1} + a_{i+1/2,j+1/2}\,u_{i+1,j+1} +$$
$$+ c_{i-1/2,j-1/2}^{i-1/2,j+1/2}\,u_{i-1,j} + d_{i,j}^{(4)}\,u_{i,j} + c_{i+1/2,j-1/2}^{i+1/2,j+1/2}\,u_{i+1,j} +$$
$$+ a_{i-1/2,j-1/2}\,u_{i-1,j-1} + b_{i-1/2,j-1/2}^{i+1/2,j-1/2}\,u_{i,j-1} + a_{i+1/2,j-1/2}\,u_{i+1,j-1} =$$
$$= \frac{3}{4}(f_{i,j+1} + f_{i+1,j+1} + f_{i-1,j} + f_{i+1,j} + f_{i-1,j-1} + f_{i,j-1} + 2f_{i,j}).$$
$$(i = \overline{1, I-1}\,;\ j = 1, J-1)$$

where

$$a = -h^{-2}p - k^{-2}q + \frac{1}{6}r,\ b = h^{-2}p - 2k^{-2}q + \frac{1}{3}r,$$
$$c = -2h^{-2}p + k^{-2}q + \frac{1}{3}r,\ d = 2\left(h^{-2}p + k^{-2}q + \frac{1}{3}r\right),$$
$$(*)\qquad d_{i,j}^{(4)} = d_{i-1/2,j-1/2\,;\,i+1/2,j-1/2}^{i-1/2,j+1/2\,;\,i+1/2,j+1/2};$$
$$u(x_i, y_j) = g(x_i, y_j),\ (x_i, y_j) \in \Gamma_h$$

here, a convention has been used in writing: several nodes attached to one letter $(a, b, c, d)$ note the summation of the letter at those nodes; e.g. the symbol in (*) stands for the sum of the values of $d$ ($d$ function) at the four fractionary nodes.

*b). The Helmholtz equation with constant coefficients.* Let us assume that the functions $p$, $q$ and $r$ get reduced to real constants. We note:

$$\beta = \frac{2}{H}\left(2qh^2 - pk^2 - \frac{1}{3}rh^2k^2\right),\ \gamma = \frac{2}{H}\left(qh^2 - 2pk^2 + \frac{1}{3}rh^2k^2\right)$$
$$(13)$$
$$\lambda = \frac{1}{H}\left(qh^2 + pk^2 + \frac{1}{3}rh^2k^2\right),\ K = \frac{h^2k^2}{H},\ H = qh^2 + pk^2 - \frac{1}{6}rh^2k^2$$

From (12) we obtain the nine node finite difference equation for the Helmholtz equation with constant coefficients:

$$(14)\qquad \Delta_9\{u_{i,j}\} \equiv u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j-1} + u_{i+1,j-1} +$$
$$+ \beta(u_{i,j-1} + u_{i,j+1}) + \gamma(u_{i-1,j} + u_{i+1,j}) - 8\lambda u_{i,j} =$$
$$= -\frac{3}{4}K(f_{i,j+1} + f_{i+1,j+1} + f_{i-1,j} + 2f_{i,j} + f_{i+1,j} + f_{i-1,j-1} + f_{i,j-1});$$
$$u(x_i, y_j) = g(x_i, y_j),\ (x_i, y_j) \in \Gamma_h$$

The equation (14) can be written in the form of the stencil:

$$(15)\qquad \begin{array}{|c|c|c|}\hline 1 & \beta & 1 \\ \hline \gamma & -8\lambda & \gamma \\ \hline 1 & \beta & 1 \\ \hline \end{array}\; u(ih, jk) = -\frac{3}{4} K \begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline 1 & 2 & 1 \\ \hline 1 & 1 & 0 \\ \hline \end{array}\; f(ih, jk)$$

where, for example, the mesh in which $-8\lambda$ is located corresponds to the point $(i, j)$ while the mesh with $\gamma$ (on the left) corresponds to the point $(i-1, j)$, that is, to the point of coordinates $((i-1)h, jk)$.

c). *The truncation error of equation (14).* Let us rewrite (14) in the form

$$\Delta_9\{u_{i,j}\} = -\frac{3}{4} K \Delta_7\{f_{i,j}\}$$

where $\{\cdot\}$ represents the grid function, whereas $\Delta$ represents the finite difference operator on the nine node and respectively seven node stencils, shown in (15).

Returning to the differential equation (1) we let $u(x, y)$ be the value of the exact solution at the point $(x, y)$ and $u(x_i, y_j) = u_{i,j}$. The truncation error $\tau(x_i, y_j)$ of the scheme (14) at $(x_i, y_j)$ is by definition

$$\tau(x_i, y_j) = \Delta_9\{u(x_i, y_j)\}_{(i,j)} + \frac{3}{4} K \Delta_7\{f_{i,j}\}_{(i,j)}$$

$$\text{with } Au(x_i, y_j) - f(x_i, y_j) = 0$$

In order to express the nodal values from the operators $\Delta_9$ and $\Delta_7$ with the help of the values at $(x_i, y_j)$ the Taylor series expansion of two variables is used, in the neighbourhood of node $(x_i, y_j)$. The following evaluation is obtained

$$\tau(x_i, y_j) = -\frac{6h^2k^2}{H}\left(-p\frac{\partial^2 u}{\partial x^2} - q\frac{\partial^2 u}{\partial y^2} + ru\right)_{(i,j)} +$$

$$+\frac{h^2k^2}{2H}[h^2(D_x^4 u - 2rD_x^2 u) + k^2(qD_y^4 u - 2rD_y^2 u) + 2HD^4_{x^2y^2}u]_{(i,j)} +$$

$$+\frac{3}{2}\frac{h^2k^2}{2H}[8f + 2(h^2D_x^2 f + k^2 D_y^2 f + hkD_{xy}^2 f]_{(i,j)} + o(h^6 + k^6)$$

$$(D_x^2 u = \partial^2 u/\partial x^2,\ D_{xy}^2 f = \partial^2 f/\partial x \partial y, \text{ etc.})$$

Subsequently, the formula

$$\tau(x_i, y_j) = \frac{h^2k^2}{2H}(a_1h^2 + a_2k^2 + a_{12}hk)_{(i,j)} + o(h^6 + k^6)$$

is inferred ($a_i$ being read from the previous formula) which leads to:



If the coefficients $a_1$, $a_2$, $a_{12}$ (which contain $f$ as well as the derivatives of $u$ and $f$) are bounded functions ($u$ and $f$ are sufficiently smooth functions) we have

$$\lim_{h,k\to 0} \tau(x_i, y_j) = 0 \text{ and } \tau(x_i, y_j) = o(h^4 + k^4)$$

that is, scheme (14) is consistent with the Helmholtz partial differential equation (1) with constant coefficients, and approximates this equation with a fourth order error.

***d). Example. Equation (14) test.*** **Let us consider the laminar motion of a viscous incompressible fluid inside an $\Omega$ rectangular section channel (placed in *Oxy*). This section is bounded by the $\Gamma$ contour, and in it the velocity $u(x, y)$ of the flow verifies the equation and the Dirichlet condition:**

$$\text{(a)}\qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -f_0,\ (x, y) \in \Omega = (0, a) \times (0, b)$$

$$\text{(b)}\qquad u(x, y) = 0,\ (x, y) \in \Gamma$$

$$\left(f_0 = \frac{\Delta p}{l\mu} = \text{const} > 0,\ \Delta p = p_0 - p_l;\ l > 0, \mu > 0\right)$$

where $\Delta p > 0$ is the pressure difference on the $l$ channel length while $\mu$ is the dynamic viscosity coefficient.

Let us use scheme (14) for two approximations of the problem (a).

1. We choose $h = a/2$, $k = b/2$. In this case we have

$$p = q = 1,\ r = 0,\ K = \frac{a^2b^2}{4(a^2 + b^2)}$$

and using the condition (b) scheme is reduced to the equality

$$u_{1,1} = \frac{3}{16}\frac{a^2b^2}{a^2 + b^2} f_0 \text{ and } u_{1,1} = 0{,}09375\, a^2 f_0 \text{ if } a = b$$

which represents the approximate value of the velocity in the centre of the rectangle and of the square, respectively. For $a = b = 2$, $f_0 = 1$ we get $u_{1,1} \equiv U_c = 0{,}375$ (in the units of $a$ and $f_0$; $[f_0] = L^{-1}T^{-1}$).

**2. We choose $h = k = a/4$ and the square section $\Omega = (0, a) \times (0, a)$. Then due to the symmetry of the motion, only the $u_{2,2}$, $u_{2,3}$ and $u_{3,3}$ values will be unknown on the grid $\Omega_h$. We have $\beta = \gamma = \lambda = 1$ and $K = h^2/2 = a^2/32$. Equation (14) leads to the linear and symmetrical system:**

$$-2u_{2,2} + u_{2,3} + u_{3,3} = -\frac{3}{64}a^2 f_0$$

$$u_{2,2} - 6u_{2,3} + 2u_{3,3} = -\frac{3}{16}a^2 f_0$$

$$u_{2,2} + 2u_{2,3} - 8u_{3,3} = -\frac{3}{16}a^2 f_0$$

which, using the Gauss elimination method ($m = 3a^2 f_0/16$) :

$$\left[\begin{array}{rrr|r} \boxed{-2} & 1 & 1 & -m/4 \\ 1 & -6 & 2 & -m \\ 1 & 2 & -8 & -m \end{array}\right] \overset{(1)}{\Rightarrow} \left[\begin{array}{rr|r} \boxed{11} & -5 & 9m/4 \\ -5 & 15 & 9m/4 \end{array}\right] \overset{(2)}{\Rightarrow} [140/36m]$$

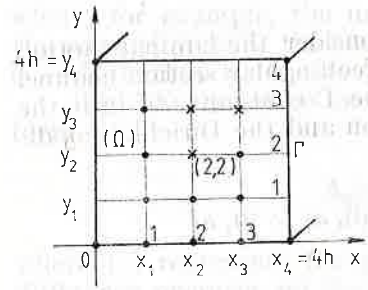


Fig. 2

has the solution

$$u_{3,3} = \frac{36m}{140} = 0{,}04821 a^2 f_0 ;$$

$$u_{2,3} = 0{,}06027 a^2 f_0 ;$$

$$u_{2,2} = 0{,}07768 a^2 f_0$$

Let us compare the central value $u_{2,2}$ with the one obtainable by applying the Ritz global method with approximate solution $U \in D(A) = \{U | U \in C^2(\Omega),\ U|_\Gamma = 0\}$ of the form

$$U(x, y) = cxy(x - a)(y - a), \quad 0 \leqslant x, y \leqslant a$$

where the constant $c$ is determined by means of the minimum condition of the functional, inferred from (3),

$$F(u) = \iint_\Omega [(\nabla u)^2 - 2 f_0 u]\, dx\, dy, \text{ for } u = U$$

In the centre of the square we obtain the value $U_c = 5a^2 f_0/64 = 0{,}078125 \cdot a^2 f_0$ which is in agreement with the value $u_{2,2}$ given by scheme (14).

*Remark.* The value $U_c$ is also obtained in [2] by means of the Galerkin method and may be found in the schemes in [1].

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