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APPLICATION OF SOME METHODS OF APPROXIMATION THEORY

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Abstract. We establish inequalities for powers of polynomials in several metrics by applying Markov-Bernstein's type theorems. The results are extended to some classes of regular functions which (from Jackson's theorem) are well approximated by polynomials.

Introduction. Let f be a real-valued function in $C^p([-1, 1])$ and $m(x) = \prod_{i=1}^{k} |x - a_i|^{\sigma_i}$ where $(a_1, a_2, \ldots, a_k)$ is a sequence of (not necessarily distinct) points in $[-1, 1]$ and $\sigma_1, \ldots, \sigma_k$ are real positive exponents. The purpose of this paper is to prove that one can give a lower bound for $\sup_{|x|\leq 1} |f(x)m(x)|$ depending only on $f, f^{(p)}$ and $\sum_{i=1}^{k} \sigma_i$ but not on the positions of the a_i' s in $[-1, 1]$.

We use here methods coming from approximation theory: in the first part, we prove the result for polynomials by means of Markov's inequality; in the second part the result is extended to functions in $C^p([-1,1])$ which from Jackson's theorem can be approximated up to a well-controlled error, by polynomials.

Notation. We denote by H_n the set of polynomials of degree n or less. For any continuous function f on $[-1, 1]$, we set $\|f(x)\| = \|f\| = \sup_{|x|\leq 1} |f(x)|$.

1. The aim of this first part is to establish the two following theorems:

THEOREM 1. *For $a_1, \ldots, a_k$ real, for $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$, δ real satisfying $0 \leq \alpha_i \leq \beta_i$, $\beta_i \neq 0$ $(i = 1, \ldots, k)$, $\delta \geq 0$, and for any $P \in H_n$ we have*

$$\left\| |P(x)|^{\delta} \prod_{i=1}^{k} |x - a_i|^{\alpha_i} \right\| \leq$$

$$\leq e^{2k} \left(n\delta + \sum_{i=1}^{k} \beta_i \right)^{2 \sum_{i=1}^{k} (\beta_i - \alpha_i)} \left\| |P(x)|^{\delta} \prod_{i=1}^{k} |x - a_i|^{\beta_i} \right\|.$$

THEOREM 2. *Let* l *be such that* $0 \leqslant l < 1$.

* *For* $a_1, \ldots, a_k$ *real satisfying* $|a_i| \leqslant l$ $(i = 1, \ldots, k)$;

* *for* $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, \delta$ *real satisfying* $0 \leqslant \alpha_i \leqslant \beta_i, \beta_i \neq 0$ $(i = 1, \ldots, k)$, $\delta \geqslant 0$,

* *for any* $P \in H_n$, *we have*

$$\left\| |P(x)|^\delta \prod_{i=1}^{k} |x - a_i|^{\alpha_i} \right\| \leqslant$$

$$e^k \left[2(1 - l)^{-1} \left(n\delta + \sum_{i=1}^{k} \beta_i \right) \right]^{2 \sum_{i=1}^{k} (\beta_i - \alpha_i)} \left\| |P(x)|^\delta \prod_{i=1}^{k} |x - a_i|^{\beta_i} \right\|.$$

Remark. We note the fact that the result is quite independant of the (not necessarily distinct) positions of the a_i's in either R (first theorem) or in $[-l, l]$ (second theorem).

Before proving Theorems 1 and 2 we establish some lemmas.

LEMMA 1. *For any polynomial* $P \in H_n$ *and any* $p \in \mathbb{N}$ *we have*

$$\|P^{(p)}\| \leqslant n^{2p}(1/p\,!)\|P\|.$$

Proof. We have [1, p. 141]:

$$\|P^{(p)}\| \leqslant \frac{n^2(n^2-1)\ldots(n^2-(p-1)^2)}{1.3.5\ldots(2p-1)}\|P\|$$

whence the lemma follows immediately.

LEMMA 2. *For any polynomial* $P \in H_n$, *any* $a \in \mathbb{R}$ *and any* $p \in \mathbb{N}$ *we have* $\|P\| \leqslant (n + p)^{2p}(1/p\,!)^2 \|(x - a)^p P(x)\|$.

Proof. First we assume $|a| \leqslant 1$ and we put $R(x) = (x - a)^p P(x)$. Applying Taylor's formula up to order p gives:

$$(x - a)^p P(x) = R(x) = (1/p\,!)(x - a)^p R^{(p)}(b)$$

for some b between x and a. Therefore, $P(x) = (1/p\,!)R^{(p)}(b)$ and, using Lemma 1:

$$\|P\| \leqslant (1/p\,!)\|R^{(p)}\| \leqslant (1/p\,!)^2(n + p)^{2p}\|R\|.$$

Now let us assume $|a| > 1$. Clearly we have:

$$\|(x - a)^p P(x)\| \geqslant \begin{cases} \|(x - 1)^p P(x)\| & \text{if } a > 0 \\ \|(x + 1)^p P(x)\| & \text{if } a < 0 \end{cases}$$

which leads us to the case $|a| \leqslant 1$.

LEMMA 3. *For any polynomial* $P \in H_n$, *any* $p \in \mathbb{N}$ *and any* $x \in]-1, 1[$ *we have* $|P^{(p)}(x)| \leqslant (n/(1 - |x|))^p \|P\|$.

Proof. From [2, p. 227], for any $P \in H_n$ and any interval $[c, d]$ we have $|P^{(p)}((c + d)/2)| \leqslant (2n/(d - c))^p \sup_{x \in [c,d]} |P(x)|$. Let $x \in]-1, 1[$. If $x \geqslant 0$ the interval $[2x - 1, 1]$ is included in $[-1, 1]$ and has his centre at x. Hence
$|P^{(p)}(x)| \leqslant (n/(1 - x))^p \sup_{y \in [2x-1, 1]} |P(y)| \leqslant (n/(1 - |x|))^p \|P\|$.
We have an analogous proof when $x \leqslant 0$.

LEMMA 4. *Let* l *be such that* $0 \leqslant l < 1$. *For any* $P \in H_n$, *any* a *satisfying* $|a| \leqslant l$ *and any* $p \in N$ *we have*

$$\|P\| \leqslant (1/p\,!)\ (2/(1 - l))^p (n + p)^p \|(x - a)^p P(x)\|.$$

Proof. Let us assume $a \geqslant 0$. We set $R(x) = (x - a)^p P(x)$. Applying Taylor's formula up to order p yelds $(x - a)^p P(x) = R(x) = (1/p\,!)(x - a)^p R'(b)$ for some b between x and a, therefore $P(x) = (1/p\,!)R'(b)$.
If $x \in [a - (1 - a)/2, a + (1 - a)/2] = [(a - 1)/2, (a + 1)/2]$, b belongs to the same interval and, from Lemma 3

$$|P(x)| \leqslant (1/p\,!)\ [(n + p)/(1 - (a + 1)/2)]^p \|R\|$$

$$= (1/p\,!)\ [2(n + p)/(1 - a)]^p \|R\|.$$

If $x \notin [(a - 1)/2, (a + 1)/2]$, $|x - a| > (1 - a)/2$ and

$$|P(x)| \leqslant [2/(1 - a)]^p |x - a|^p |P(x)| \leqslant [2/(1 - a)]^p \|R\|.$$

Clearly, $[2/(1 - a)]^p \leqslant (1/p\,!)[2(n + p)/(1 - a)]^p$ therefore

$$\|P\| \leqslant (1/p\,!)[2(n + p)/(1 - a)]^p \|(x - a)^p P(x)\|.$$

To complete the proof we observe that $1/(1 - a) \leqslant 1/(1 - l)$. We have an analogous proof when $a \leqslant 0$.

Proof of Theorem 1. First we remark that for any given $a_1, \ldots, a_j$, P, the function

$$(\gamma_1, \ldots, \gamma_j, \delta) \to \| |P(x)|\, |x - a_1|^{\gamma_1} \ldots |x - a_j|^{\gamma_j} \|$$

is continuous from $(]0, +\infty[)^{j+1}$ to R^+. It is then sufficient to prove the theorem when $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, \delta$ are rational. From now on we will suppose this to be the case.

One can find an integer t such that $t\delta, t\alpha_i, t\beta_i$ $(i = 1, \ldots, k)$ are integers. We put

$$N_1 = \| |P(x)|^\delta \prod_{i=1}^{k} |x - a_i|^{\alpha_i} \| = \|(P(x))^{t\delta} \prod_{i=1}^{k} (x - a_i)^{t\alpha_i} \|^{1/t}$$

$$N_2 = \| |P(x)|^\delta \prod_{i=1}^{k} |x - a_i|^{\beta_i} \| = \|(P(x))^{t\delta} \prod_{i=1}^{k} (x - a_i)^{t\beta_i} \|^{1/t}.$$

Then applying Lemma 2 k-times gives:

$$N_1 \leqslant \left[\prod_{i=1}^{k} \frac{\left[t \left(n\delta + \sum_{i=1}^{k} \beta_i \right) \right]^{2t(\beta_i - \alpha_i)}}{((t\alpha_i - t\beta_i)\,!)^2} \right]^{1/t} N_2.$$

Or, setting $t(\beta_i - \alpha_i) = d_i$,

$$N_1 \leqslant \left(n\delta + \sum_{i=1}^{k} \beta_i\right)^{2\sum_{i=1}^{k}(\beta_i - \alpha_i)} \cdot \prod_{i=1}^{k} ((t^2)^{d_i}/(d_i\,!)^2/(d_i\,!)^2)^{1/t} N_2.$$

Therefore to prove Theorem 1 it is sufficient to observe that

$$((t^2)^{d_i}/(d_i\,!)^2)^{1/t} = ((t^{d_i}/d_i\,!)\,(t^{d_i}/d_i\,!))^{1/t} \leqslant (e^t \cdot e^t)^{1/t} = e^2.$$

Proof of Theorem 2. We carry out the same proof as in Theorem 1 but using Lemma 4 instead of Lemma 2.

2. Let $a_1, \ldots, a_k$ be (not necessarily distinct) points in $[-1, 1]$ and $\sigma_1, \ldots, \sigma_k$ be real and positive. We put $\sigma = \sum_{i=1}^{k} \sigma_i$ and $m(x) = \prod_{i=1}^{k} |x - a_i|^{\sigma_i}$.

We are going to use Theorems 1 and 2 to prove the following result :

THEOREM 3. *For each $p \in N^*$ there exist two positive constants A and B (depending in σ and k) such that, for any function $f \in C^p([-1, 1])$ we have*

either $$\|fm\| \geqslant A\|f\|$$

or $$\|fm\| \geqslant B\,\|f\|^{1+(2\sigma/(p-2\sigma))}\,\|f^{(p)}\|^{-2\sigma/(p-2\sigma)}.$$

Furthermore if there exists l satisfying $0 \leqslant l < 1$ such that $|a_i| \leqslant l$ $(i = 1, \ldots, k)$ then σ can be replaced by $\sigma/2$ in the second inequality.

Remark. The constants A and B depend neither on the individual σ_i's (but only on their sum and number) nor on the a_i's positions in $[-1, 1]$ and $[-l, l]$.

Proof. From Jackson's theorem (see [1, *p.* 128]) there exists a constant C_1 such that, for any n satisfying $n > p$, one can find $P \in H_n$ such that $\|f - P\| \leqslant C_1 n^{-p}\|f^{(p)}\|$. Therefore

(1) $$\|f\| \leqslant \|f - P\| + \|P\| \leqslant C_1 n^{-p}\|f^{(p)}\| + \|P\|.$$

But applying Theorem 1 with $\delta = 1$, $\alpha_i = 0$, $\beta_i = \sigma_i$ $(i = 1, \ldots, k)$ yelds $\|P\| \leqslant e^{2k}(n + \sigma)^{2\sigma}\|Pm\| \leqslant e^{2k} n^{2\sigma}(1 + \sigma/p)^{2\sigma}\|Pm\|$,

(2) $$\text{i.e. } \|P\| \leqslant C_2 n^{2\sigma}\|Pm\|.$$

On the other hand, $\|Pm\| \leqslant \|Pm - fm\| + \|fm\|$

$$\leqslant \|m\|\,\|P - f\| + \|fm\|$$

Therefore, since $\|m\| \leqslant 2^{\sigma}$

(3) $$\|Pm\| \leqslant C_3 n^{-p}\|f^{(p)}\| + \|fm\|$$

From (1), (2) and (3) we deduce the existence of a constant C_4 such that, for any integer n strictly grater than p,

(4) $$\|f\| \leqslant C_4 n^{2\sigma}(n^{-p}\|f^{(p)}\| + \|fm\|).$$

We will now consider two cases separately.

Firstly, let us assume that $(\|f^{(p)}\|/\|fm\|)^{1/p} \leqslant p$. Then, $\|f^{(p)}\| \leqslant \leqslant p^p\|fm\|$ which implies, by (4) with $n = p + 1$: $\|fm\| \geqslant A\|f\|$. Now, let us assume that $(\|f^{(p)}\|/\|fm\|)^{1/p} > p$.
Let n an integer satisfying

$$(\|f^{(p)}\|/\|fm\|)^{1/p} \leqslant n < (\|f^{(p)}\|/\|fm\|)^{1/p} + 1.$$

Then we have $n < 2(\|f^{(p)}\|/\|fm\|)^{1/p}$ and $n^{-p}\|f^{(p)}\| \leqslant \|fm\|$. Substituting in (4) yelds

$$\|f\| \leqslant C_5\|f^{(p)}\|^{2\sigma/p}\|fm\|^{1-(2\sigma/p)}, \text{ therefore}$$

$$\|fm\| \geqslant \|f\|^{1+(2\sigma/(p-2\sigma))}\|f^{(p)}\|^{-2\sigma/(p-2\sigma)}.$$

If the a_i's belongs to $[-l, l]$, the proof is carried out in the same way, using Theorem 2 instead of Theorem 1.

Remark. Let $E = C^p([-1, 1])$ with the norm $\|\|f_E = \|f^{(p)}\| + \|fm\|$ and $F = C^p[(-1,1])$ with the norm $\|f\|_F = \|f\|$. Inequality (4) then says that the inclusion map of E in F is continuous.

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