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SOME REMARKS ON A THEOREM ON BEST APPROXIMATIONS

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Abstract. In this paper we give a short proof of the Theorem from [12]. Some generalizations of this theorem in normed, convex metric and topological vector spaces will be obtained.

1. The following result, proved by Ky Fan, is well known [3].

LEMMA 1. *Let C be a compact convex subset of a normed space E, and let $F : C \to E$ be continuous. Then there exists at least one $y_0 \in C$ such that :*

$$\|y_0 - F(y_0)\| = \inf_{x \in C} \|x - F(y_0)\|.$$

The proof of this lemma is given by the method of KKM mapping [2].

Definition 1. *Let E be a vector space and X be an arbitrary subset of E. A function $G : X \to 2^E$ is called a Knaster-Kuratowski-Mazurkiewicz map (or simply a* KKM*-map) provided :*

$$\mathrm{co}\,\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^{n} G(x_i)$$

for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X.

The following theorem is used in the proof of Lemma 1.

THEOREM 1. (Ky Fan) *Let E be a topological vector space, $X \subseteq E$ and $G : X \to 2^E$ a* KKM *mapping. If the sets $G(x)$ are closed, for every $x \in X$, and if there exists $x_0 \in X$ such that $G(x_0)$ is compact then $\bigcap_{x \in X} G(x) \neq \emptyset$.*

In [12] a generalization of Lemma 1 is proved and, in the proof, the author used Bohnenblust's and Karlin'sfixed point theorem for set-valued mappings. Using the method of KKM mapping we shall generalize Prollas result. Let $(E, \|\ \|)$ be a Banach space, M a convex subset of E and $g : M \to E$. The mapping g is said to be *almost affine* if [12] :

$$\|g(t) - y\| \leqslant \lambda \|g(t_1) - y\| + (1 - \lambda)\|g(t_2) - y\|$$

for all $t_1, t_2 \in M$ and all $\lambda \in (0, 1)$ where y is an arbitrary element of E and $t = \lambda t_1 + (1 - \lambda)t_2$.

LEMMA 2. *If E, M and g are as above and g is almost affine then:*

$$\left\| g\left(\sum_{i=1}^{n} \lambda_i t_i\right) - y \right\| \leqslant \sum_{i=1}^{n} \lambda_i \|g(t_i) - y\| \tag{1}$$

for every $n \in N$ and every $\lambda_i \geqslant 0 (i \in \{1, 2, \ldots, n\})$, $\sum_{i=1}^{n} \lambda_i = 1$ every $t_i \in M$ $(i \in \{1, 2, \ldots, n\})$ and every $y \in E$.

Proof. For $n = 2$ (1) is satisfied by the definition of an almost affine mapping. Suppose that for every $\lambda_i \geqslant 0$ $(i \in \{1, 2, \ldots, n-1\})$ such that $\sum_{i=1}^{n-1} \lambda_i = 1$ we have:

$$\left\| g\left(\sum_{i=1}^{n-1} \lambda_i t_i\right) - y \right\| \leqslant \sum_{i=1}^{n-1} \lambda_i \|g(t_i) - y\| \tag{2}$$

for every $t_i \in M$ $(i \in \{1, 2, \ldots, n-1\})$ and every $y \in E$. Let us prove that (1) holds. Let $x_i \in M$, $\mu_i \geqslant 0$ $(i \in \{1, 2, \ldots, n\})$ and $\sum_{i=1}^{n} \mu_i = 1$. Then $x = \sum_{i=1}^{n-1} \frac{\mu_i}{1-\mu_i} x_i \in M$ since $\sum_{i=1}^{n-1} \frac{\mu_i}{1-\mu_i} = 1$ and M is convex. Since g is almost affine we have that (2) implies:

$$\left\| g\left(\sum_{i=1}^{n} \mu_i x_i\right) - y \right\| = \|g((1-\mu_n)x + \mu_n x_n) - y\| \leqslant$$

$$\leqslant (1-\mu_n)\|g(x) - y\| + \mu_n\|g(x_n) - y\| = (1-\mu_n)\left\| g\left(\sum_{i=1}^{n-1} \frac{\mu_i}{1-\mu_n} x_i\right) - y \right\| +$$

$$+ \mu_n\|g(x_n) - y\| \leqslant (1-\mu_n)\left[\sum_{i=1}^{n-1} \frac{\mu_i}{1-\mu_n} \|g(x_i) - y\|\right] + \mu_n\|g(x_n) - y\| =$$

$$= \sum_{i=1}^{n} \mu_i \|g(x_i) - y\|.$$

2. The following theorem is proved in [12] and we shall give a new proof of it using Theorem 1.

THEOREM 2. *Let M be a compact and convex non-empty subset of a normed space E and let g be a continuous almost affine selfmap of M onto M. For each continuous mapping $f: M \to E$, there exists some $y_0 \in M$ such that:*

$$\|g(y_0) - f(y_0)\| = \text{dist}\,(f(y_0); M)$$

where $\text{dist}\,(f(y_0); M) = \inf_{m \in M} \|m - f(y_0)\|$.

Proof. The proof is similar to the proof of Lemma 1, given in [2]. Define the mapping $G: M \to 2^E$ in the following way:

$$G(x) = \{y \mid y \in M, \|g(y) - f(y)\| \leqslant \|g(x) - f(y)\|\}. \tag{3}$$

Since f and g are continuous it follows that $G(x)$ is closed in M and from the compactness of M is follows the compactness of $G(x)$, for every $x \in M$. Let us prove that G is a KKM mapping. If we suppose that G is not a KKM mapping then there exists a finite set $\{x_1, x_2, \ldots, x_n\} \subseteq M$ and an element y from co $\{x_1, x_2, \ldots, x_n\}$ so that $y \notin \bigcup_{i=1}^{n} G(x_i)$. From (3) we obtain that:

$$\|g(y) - f(y)\| > \|g(x_i) - f(y)\|, i \in \{1, 2, \ldots, n\}. \tag{4}$$

Suppose that $y = \sum_{i=1}^{n} \lambda_i x_i$, $\lambda_i \geqslant 0 (i \in \{1, 2, \ldots, n\})$ $\sum_{i=1}^{n} \lambda_i = 1$.

Then (4) implies that:

$$\|g(y) - f(y)\| > \sum_{i=1}^{n} \lambda_i \|g(x_i) - f(y)\|$$

and from Lemma 2 we obtain that:

$$\|g(y) - f(y)\| > \left\| g\left(\sum_{i=1}^{n} \lambda_i x_i\right) - f(y) \right\| = \|g(y) - f(y)\|$$

which is a contradiction. So, G is a KKM mapping and this implies that there exists $y_0 \in M$ such that $y_0 \in \bigcap_{x \in M} G(x)$. From (3) it follows that $\|g(y_0) - f(y_0)\| = \inf_{x \in M} \|g(x) - f(y_0)\|$. Since $g(M) = M$, we obtain that:

$$\|g(y_0) - f(y_0)\| = \text{dist}\,(f(y_0); M).$$

Using Theorem 2 we can prove the following generalization of Ky Fan fixed point theorem ([2], Theorem 2.2).

Remark. It is obvious that Theorem 2 holds if the mapping g satisfies the condition:

$$\left\| g\left(\sum_{i=1}^{n} \lambda_i x_i\right) - y \right\| \leqslant \max\{\|g(x_i) - y\|, i \in \{1, 2, \ldots, n\}\}$$

for every $x_i \in M$, $\lambda_i \geqslant 0 (i \in \{1, 2, \ldots, n\})$, $y \in E$ $\sum_{i=1}^{n} \lambda_i = 1$. A similar condition is introduced by Viorel Sadoveanu in [15].

THEOREM 3. *Let M be a compact and convex non-empty set of a normed space E, g a continuous almost affine mapping of M into M and $f: M \to E$ continuous such that for each $m \in M$ with $g(m) \neq f(m)$ the line segment $[g(m), f(m)] = \{\lambda g(m) + (1-\lambda) f(m), 0 \leqslant \lambda \leqslant 1\}$ contains at least two points of M. Then there exists $y_0 \in M$ such that:*

$$g(y_0) = f(y_0).$$

Proof. From Theorem 2 it follows that there exists $y_0 \in M$ such that $\|g(y_0) - f(y_0)\| = \text{dist}\,(f(y_0); M)$.

Suppose that $g(y_0) \neq f(y_0)$ and let $x \in M \cap [g(y_0), f(y_0)] \setminus \{g(y_0)\}$. Then $x = \lambda g(y_0) + (1-\lambda)f(y_0)$ for some $\lambda \in (0, 1)$ and so:

$$\|g(y_0) - f(y_0)\| \leqslant \|\lambda g(y_0) + (1-\lambda)f(y_0) - f(y_0)\|$$
$$= \lambda \|g(y_0) - f(y_0)\|$$

which implies that $g(y_0) = f(y_0)$.

We shall give some generalizations of Theorem 2 in metric spaces with a convex structure [16].

In 1970 Takahashi introduced the notion of a metric space with a convex structure. Some fixed point theorems in such spaces are proved in [5], [11], [13], [17].

Definition 2. Let X be a metric space and $I = [0, 1]$. A mapping $W: X \times X \times I \to X$ is said to be a convex structure on X if for all x, $y \in X$ and $\lambda \in I$ we have:

$$d(u, W(x, y, \lambda)) \leqslant \lambda d(u, x) + (1-\lambda) d(u, y)$$

for all $u \in X$. Then X, together with a convex structure, is called a convex metric space.

Definition 3. Let X be a convex metric space. A nonempty subset $K \subseteq X$ is convex if and only if:

$$W(x, y, \lambda) \in K, \textit{for every } (x, y, \lambda) \in X \times X \times [0, 1].$$

The *convex hull* $\mathrm{co}_W(A)$ of a set $A \subseteq X$ is the intersection of all convex sets containing A.

Remark. The mapping W is not continuous in general, however if X is compact that W is continuous. In a Banach space X, W is defined by $W(x, y, \lambda) = \lambda x + (1-\lambda)y$, for every $x, y \in X$ and $\lambda \in I$. If (X, d) is a linear metric space with a translation invariant metric d such that:

$$d(\lambda x + (1-\lambda)y, 0) \leqslant \lambda d(x, 0) + (1-\lambda) d(y, 0) \; (x, y \in X, \lambda \in I)$$

then X is a convex metric space.

In [7] the notion of a pseudo-convex structure is introduced.

Definition 4. Let X be a topological space and $h: X \times X \times I \to X$ so that:

(i) $h(x, y, 0) = y$, $h(x, y, 1) = x$, for every $(x, y) \in X \times X$.

(ii) For every finite subset A of X:

$h|\mathrm{co}_h(A) \times \mathrm{co}_h(A) \times I$ is continuous ($\mathrm{co}_h(A)$ is defined as $\mathrm{co}_W(A)$, for $W = h$). Then h is a pseudo-convex structure on X and (X, h) is a pseudo-convex space.

It is obvious that a convex structure W on a metric space X satisfies condition (i). If we suppose that a convex metric space (X, d) is such that $\mathrm{co}_W(A)$ is compact for every finite subset A of X then $W|\mathrm{co}_W(A) \times \mathrm{co}_W(A) \times I$ is continuous and W is a pseudo-convex structure in the sense of Definition 4. In [17] Talman introduced the notion of a strong convex structure (SCS) on a metric space. A *strong convex structure* on X is a continuous functions $K: X \times X \times X \times P \to X$, where $P = \{(t_1, t_2, t_3) \in I \times I \times I : t_1 + t_2 + t_3 = 1\}$, with the property that for each $(x_1, x_2,$

$x_3, t_1, t_2, t_3) \in X \times X \times X \times P$, $K(x_1, x_2, x_3, t_1, t_2, t_3)$ is the unique point of X which satisfies the inequality:

$$d(y, K(x_1, x_2, x_3, t_1, t_2, t_3)) \leqslant \sum_{k=1}^{3} t_k d(y, x_k)$$

for every $y \in X$. A metric space with a strong convex structure will be called *strongly convex*. By Theorem 3.1 from Talmans paper [17] it follows that for every finite set A from a strongly convex metric space X, $\mathrm{co}_{W_K}(A)$ is compact, where W_K is the induced Takahashi convex structure, defined by:

$$W_K(x_1, x_2, t) = K(x_1, x_2, t, 1-t, 0)$$

where $(x_1, x_2, t) \in X \times X \times I$. From this it follows that (X, W_K) is apseudo-convex space.

Let (X, h) be a pseudo-convex space and $R: X \to 2^X$. The mapping R is said to be an element of $\mathrm{KKM}_h(X)$ [7] if and only if for every finite subset A of X:

$$\mathrm{co}_h(A) \subseteq \bigcup_{x \in A} R(x).$$

In [7] the following proposition is proved.

PROPOSITION 1. *Let (X, d) be a complete pseudo-convex metric space and $R \in \mathrm{KKM}_h(X)$ such that $R(x)$ is closed for every $x \in X$. If for every $\varepsilon > 0$ there exists a finite set A such that: $\alpha(\bigcap_{x \in A} R(x)) < \varepsilon$ (α — the Kuratowski measure of noncompactness) then $\bigcap_{x \in A} R(x)$ is nonempty and compact.*

Definition 5. Let (X, d) be a metric space with a convex structure W, M a non-empty subset of X which is convex and $g: M \to X$. The mapping g is said to be W-almost affine if and only if for every finite set $\{x_1, x_2, \ldots, x_n\} \subseteq M$ and every $z \in X$:

$$y \in \mathrm{co}_W\{x_1, x_2, \ldots, x_n\} \Rightarrow d(g(y), z) \leqslant \max d(g(x_i), z). \; i \in \{1, 2, \ldots, n\}.$$

Using the above Proposition and Definition 5 we can prove the following generalization of Theorem 2.

THEOREM 2'. *Let M be a compact and convex non-empty subset of a convex metric space (E, d) with convex structure W, g a continuous W-almost affine mapping of M onto M and $f: M \to E$ a continuous mapping. If $\mathrm{co}_W(A)$ is compact, for every finite subset A of E, then there exists $y_0 \in M$ such that:*

$$d(g(y_0), f(y_0)) = \mathrm{dist}(f(y_0); M) \tag{5}$$

Proof. As in the proof of Theorem 2 let:

$$G(x) = \{y \mid y \in M, d(g(y), f(y)) \leqslant d(g(x), f(y))\}, \; x \in M$$

Since M is compact it is complete and $G(x)$ is compact, for every $x \in M$. So we have that $\alpha(\bigcap_{x \in A} G(x)) = 0$, for every $A \subseteq M$. Since g is W-almost affine, similarly as in Theorem 2, we conclude that $G \in \mathrm{KKM}_W(M)$ and so, using Proposition 1 it follows that $\bigcap_{x \in M} G(x) \neq \emptyset$. The rest of the proof is as in Theorem 2.

COROLLARY. *Let E be a strongly convex metric space with a strong convex structure K, M a compact, convex and non-empty subset of E, g a continuous W_K-almost affine mapping of M onto M and $f : M \to E$ a continuous mapping. Then there exists $y_0 \in M$ such that (5) is satisfied.*

THEOREM 3'. *Let M be a compact, convex and non-empty subset of a convex metric space (E, d) with convex structure W, g a continuous W-almost affine mapping of M onto M, $f : M \to E$ a continuous mapping and for each $m \in M$ such that $g(m) \neq f(m)$:*

$$\text{card}\,(\{W(g(m), f(m), \lambda) \mid \lambda \in [0, 1]\} \cap M) \geqslant 2.$$

If $\text{co}_W(A)$ is compact for every finite subset A of E then there exists $y_0 \in M$ such that $g(y_0) = f(y_0)$.

Proof. Let $y_0 \in M$ be such that $d(g(y_0), f(y_0)) = \text{dist}\,(f(y_0); M)$. If $g(y_0) \neq f(y_0)$ then there exists $x \in [W(g(y_0), f(y_0), \lambda) \cap M] \setminus \{g(y_0)\}$ for some $\lambda \in (0, 1)$. Then we have:

$$d(g(y_0), f(y_0)) \leqslant d(W(g(y_0), f(y_0), \lambda), f(y_0))$$
$$\leqslant d(g(y_0), f(y_0)) + (1 - \lambda) d(f(y_0), f(y_0)) = \lambda d(g(y_0), f(y_0))$$

which implies that $g(y_0) = f(y_0)$.

Remark. Let (E, d) be a convex metric space with convex structure W, M a closed and convex subset of E, $f : M \to E$ a continuous mapping and g a continuous mapping of M onto M such that:

$$g(W(t_1, t_2, \lambda)) = W(g(t_1), g(t_2), \lambda), \text{ for every } t_1, t_2 \in M$$

and every $\lambda \in [0, 1]$. Using the same method as in Prollas paper [12] we can prove the existence of $y_0 \in M$ such that $d(g(y_0), f(y_0)) = \text{dist}\,(f(y_0); M)$ if M is such that Fix $(F) \neq \emptyset$ (the set of fixed points of F) for every multi-valued mapping $F : M \to 2^M \setminus \emptyset$ which is closed (in the sense of the closed graph) and such that $F(x)$ is closed and convex. To prove this let us define the mapping $F_f : M \to 2^M$ in the following way:

$$F_f(x) = \{m \mid m \in M, d(g(m), f(x)) \leqslant \frac{1}{2} [d(g(x), f(x)) + \text{dist}\,(f(x); M)]\}$$

for every $x \in M$.

As in [12] it follows that F_f is a closed mapping such that $F_f(x)$ is closed and non-empty, for every $m \in M$. The convexity of $F_f(x)$ follows from the inequalities:

$$d(g(W(t_1, t_2, \lambda)), f(x)) = d(W(g(t_1), g(t_2), \lambda), f(x))$$
$$\leqslant \lambda d(g(t_1), f(x)) + (1 - \lambda) d(g(t_2), f(x))$$
$$\leqslant \frac{1}{2} [d(g(x), f(x)) + \text{dist}\,(f(x); M)], \text{ for every } t_1, t_2 \in F_f(x), \lambda \in [0, 1].$$

Then for $y_0 \in$ Fix (F_f) it follows that $d(g(y_0), f(y_0)) = \text{dist}\,(f(y_0); M)$

3. Using the following result proved by Allen in [1] we can prove a generalization of Theorem 2.

PROPOSITION 2. *Let X be a nonempty, convex set in a topological vector space and $f : X \times X \to \mathbb{R}$ such that:*

(a) For each fixed $x \in X$, $f(x, y)$ is a lower semicontinuous functions of y on X.

(b) For each fixed $y \in X$, $f(x, y)$ is a quasi-concave function of x on X.

(c) $f(x, x) \leqslant 0$, for all $x \in X$.

(d) X has a nonempty compact convex subset X_0 such that the set $\{y \mid y \in X, f(x, y) \leqslant 0$ for all $x \in X_0\}$ is compact. Then there exists a point $\hat{y} \in X$ such that $f(x, \hat{y}) \leqslant 0$, for all $x \in X$.

THEOREM 4. *Let M be a non-empty convex set in a normed vector space E, $h : M \to E$ a continuous mapping, $g : M \to E$ almost affine continuous mapping, M_0 a nonempty compact convex subset of M and K a nonempty compact subset of M. If for every $x \in M \setminus K$ there is a point $x \in M_0$ such that:*

$$\|g(x) - h(y)\| < \|g(y) - h(y)\|$$

then there exists $\hat{y} \in K$ so that:

$$\|g(\hat{y}) - h(\hat{y})\| = \min \|g(x) - h(\hat{y})\|. \tag{6}$$

Proof. Let $f : M \times M \to R$ be defined in the following way:

$$f(x, y) = \|g(y) - h(y)\| - \|g(x) - h(y)\| \quad (x, y \in M).$$

Let us prove that the set

$$\{y \mid y \in M, f(x, y) \leqslant 0, \text{ for all } x \in M_0\}$$

is compact. Since for every $y \in M \setminus K$ there exists $x \in M_0$ such that $f(x, y) > 0$, it follows that

$$A = \{y \mid y \in M, f(x, y) \leqslant 0, \text{ for all } x \in M_0\} \subseteq K.$$

The set A is closed and so from the compactness of K it follows that A is compact. Let us prove that for each $y \in X$, $f(x, y)$ is a quasi-concave function. Let $t \in R$, $y \in M$, $x_i \in \{x \mid x \in M, f(x, y) > t\}$, $i \in \{1, 2\}$ $u, v \geqslant 0$, $u + v = 1$.

Then $\|g(x_i) - h(y)\| < \|g(y) - h(y)\| - t$, $i \in \{1, 2\}$; and so:

$$\|g(ux_1 + vx_2) - h(y)\| \leqslant u\|g(x_1) - h(y)\| + v\|g(x_2) - h(y)\|$$

$< \|g(y) - h(y)\| - t$. Hence, all the conditions (a) – (d) are satisfied.

If $\hat{y} \in K$ is such that $f(x, \hat{y}) \leqslant 0$, for all $x \in M$, then (6) is satisfied.

Remark. Theorem 4 can be generalized to topological vector spaces. Let E be a topological vector space, M a convex subset of E and $g : M \to E$. The mapping g is said to be almost affine if for every continuous seminorm p on E, every $x_1, x_2 \in M$, every $z \in E$ and every $u, v \geqslant 0$, $u + v = 1$ we have:

$$p(g(ux_1 + vx_2) - z) \leqslant up(g(x_1) - z) + vp(g(x_2) - z).$$

Using this definition Theorem 4 can be generalized in the following way: *Let M be a non-empty convex set in a topological vector space E, $h : M \to E$ a continuous mapping, $g : M \to E$ almost affine continuous mapping, M_0 a nonempty compact convex subset of M and K a non-empty compact subset of M. If p is a continuous seminorm on E such that for every $y \in M \setminus K$ there is a point $x \in M_0$ such that $p(g(x) - h(y)) < p(g(y) - h(y))$ then there exists $\hat{y}_p \in K$ so that:*

$$(7) \qquad p(g(\hat{y}_p) - h(\hat{y}_p)) = \min_{x \in M} p(g(x) - h(\hat{y}_p)).$$

If E is a topological vector space then E has sufficiently many continuous linear functionals if for every $x \neq 0$, $x \in E$ there exists a continuous linear functional f such that $f(x) \neq 0$.

Similarly as in [8] we shall prove the following theorem using the above result on best approximations.

THEOREM 5. *Let M be a nonempty, convex subset of a topological vector space E which has sufficiently many continuous linear functionals, $h : M \to E$ a continuous mapping, g an almost affine continuous mapping of M onto M, M_0 a nonempty compact convex subset of M and K a nonempty compact subset of M. Suppose that the following two conditions are satisfied:*

(i) If p is a continuous seminorm on E such that $p(z) > 0$ for every $z \in B = h(K) - g(K)$ then for every $y \in M \setminus K$ there exists $x_p \in M_0$ such that:

$$p(g(x_p) - h(y)) < p(g(y) - h(y)).$$

(ii) For every $y \in M$, $g(y) \neq h(y)$ implies that there exists $t \in (-1, 1)$ such that $tg(y) + (1 - t)h(y) \in M$.

Then there exists $y_0 \in K$ such that $g(y_0) = h(y_0)$.

Proof. Suppose that $g(y) \neq h(y)$, for every $y \in K$. Then $0 \notin B$ and so for every $x \in B$ there exists a continuous linear functional f_x such that $f_x(x) \neq 0$. Since f_x is continuous there exists an open neighbourhood U_x of $x \in B$ so that $f_x(y) \neq 0$ for every $y \in U_x$. If $B \subseteq \bigcup_{i=1}^{n} U_{x_i}$, let $p(u) = \sum_{i=1}^{n} |f_{x_i}(u)|$, for every $u \in E$. Then $p(z) > 0$, for every $z \in B$. From (i) and (7) it follows that there exists $\hat{y}_p \in K$ so that:

$$0 < p(g(\hat{y}_p) - h(\hat{y}_p)) \leq p(g(x) - h(\hat{y}_p)), \text{ for every } x \in M.$$

Let $t \in (-1, 1)$ be such that $t\,g(\hat{y}_p) + (1 - t)h(\hat{y}_p) \in M$. Then we have:

$$0 < p(g(\hat{y}_p) - h(\hat{y}_p)) \leq p(tg(\hat{y}_p) + (1 - t)h(\hat{y}_p) - h(\hat{y}_p)) = tp(g(\hat{y}_p) - h(\hat{y}_p))$$

which is a contradiction.

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