MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 15, N° 1, 1986, pp. 37–40

SADDLE POINT DUALITY THEOREMS FOR PARETO OPTIMIZATION

PAUL IACOB
(Braşov)

Let $X, Y \subseteq R^n$ and let

(1)
$$F : R^n \to R^p$$
$$\Phi : X \times Y \to R^p$$

be two applications.

Definition 1. A point x^0 in a subset D of R^n is called a Pareto minimum point for the function F on D if there is no $x \in D$ such that

(2)
$$F(x) \leqslant F(x^0)$$

The point x^0 is called a Pareto maximum point for F on D if there is no $x \in D$ such that

(3)
$$F(x) \geqslant F(x^0)$$

The set of all Pareto minimum points for F on D is denoted by $\mathrm{mP}(F/D)$ and that of all *maximum Pareto points* by $\mathrm{MP}(F/D)$.

In Definition 1 the order $\leqslant$ in R^p is understood in the sense that: $x \leqslant y$ iff $x_i \leqslant y_i$, $i = 1, \ldots, p$, and $x \neq y$ for $x = (x_1, \ldots, x_p)$ and $y = (y_1, \ldots, y_p)$ in R^p. The relation $x \leqq y$ means that $x_i \leqslant y_i$ for $i = 1, \ldots, p$ and $x < y$ means that $x_i < y_i$, for $i = 1, \ldots, p$.

In the following the domain D will be defined by some inequality restrictions, i.e.

(4)
$$D = \{x \in R^n : G(x) \leqq 0\}$$

where

(5)
$$G : R^n \to R^m.$$

Definition 2. A point $(x^0, y^0) \in X \times Y$ is called a Pareto saddle point for Φ if there is no point $x \in X$ such that

(6)
$$\Phi(x, y^0) \leqslant \Phi(x^0, y^0)$$

and there is no point $y \in Y$ such that

(7)
$$\Phi(x^0, y^0) \leqslant \Phi(x^0, y)$$

The set of all Pareto saddle points for Φ is denoted by $SA(\Phi/X \times Y)$.

Let us mention that this definition differs from the definition given by Drăgușin in [1].

THEOREM 1. *A point $(x^0, y^0) \in X \times Y$ is a Pareto saddle point for Φ if and only if the following conditions hold:*

1) $x^0 \in mP(\Phi(., y^0)/X)$ *and*
2) $y^0 \in MP(\Phi(x^0,.)/Y)$.

Proof. Follows by Definitions 1 and 2.

Put

$$\bar{m} = \bigcup_{y \in Y} \{(x, y) \in X \times Y : x \in mP(\Phi(.,y)/X)\}, \text{ and} \tag{8}$$

$$\bar{M} = \bigcup_{x \in X} \{(x, y) \in X \times Y : y \in mP(\Phi(x,.)/Y)\}.$$

The primal problem (problem (P)) consists in finding the Pareto minimum points of the function Φ on $\bar{M}$, i.e. of the set $mP(\Phi/\bar{M})$.

The dual problem (problem (D)) consists in finding the Pareto maximum points of the function Φ on $\bar{m}$, i.e. of the set $MP(\Phi/\bar{m})$.

In the duality theorems proved bellow we shall use the following condition:

Condition (A). We say that on the points (x^1, y^1) and (x^2, y^2) in $X \times Y$ is satisfied condition (A) if

$$\Phi(x^2, y^1) \leqq \Phi(x^2, y^2) \text{ or } \Phi(x^1, y^1) \leqq \Phi(x^2, y^1) \tag{9}$$

THEOREM 2. *Let $(x^1, y^1) \in \bar{m}$ and $(x^2, y^2) \in \bar{M}$, where $\bar{m}$, $\bar{M}$ are defined by* (8). *If the condition A is satisfied on these points then the relation*

$$\Phi(x^2, y^2) \leq \Phi(x^1, y^1)$$

does not hold.

Proof. From $(x^1, y^1) \in \bar{m}$ it follows

$$\Phi(x^2, y^1) \nleqq \Phi(x^1, y^1) \tag{10}$$

and from $(x^2, y^2) \in \bar{M}$ it follows

$$\Phi(x^2, y^2) \nleqq \Phi(x^2, y^1) \tag{11}$$

Suppose that

$$\Phi(x^2, y^2) \leq \Phi(x^1, y^1) \tag{12}$$

Write $Y = Y^1 \cup Y^2$, where Y^1 is the set of all points $y \in Y$ where $\Phi(x^2, y)$ is comparable with $\Phi(x^2, y^1)$ and $Y^2 = Y \setminus Y^1$. If $y^2 \in Y^1$ then

$$\Phi(x^2, y^1) \leqq \Phi(x^2, y^2) \tag{13}$$

But, by (12) and (13), $\Phi(x^2, y^1) \leq \Phi(x^1, y^1)$ in contradiction with (10). Therefore $y^2 \in Y^2$, so that

$$\Phi(x^2, y^2) \not\lesseqgtr \Phi(x^2, y^1). \tag{14}$$

Similarly, the set X can be written as $X = X^1 \cup X^2$, where X^1 is the set of all $x \in X$ such that $\Phi(x, y^1)$ is comparable with $\Phi(x^2, y^1)$ and $X^2 = X \setminus X^1$. If $x^1 \in X^1$ then

$$\Phi(x^1, y^1) \leqq \Phi(x^2, y^1) \tag{15}$$

and, by (12), it follows $\Phi(x^2, y^2) \leq \Phi(x^2, y^1)$, in contradiction with (11). Therefore $x^1 \in X^2$, so that

$$\Phi(x^1, y^1) \not\lesseqgtr \Phi(x^2, y^1). \tag{16}$$

But, relations (14) and (16) contradict condition (A).

Remark 1. If $\Phi : X \times Y \to R$ (i.e. in the case of the optimization with one objectiv function) then Condition (A) is fulfilled and one obtains well known duality theorems (see, e.g. [4]).

In Theorem 7 bellow we shall give an important case when condition (A) holds for every pair of points (x^1, y^1) $\bar{m}$ and $(x^2, y^2) \in \bar{M}$.

From Theorem 2 one obtains

THEOREM 3. *If condition* (A) *holds then*

$$\Phi(x^2, y^2) \nleqq \Phi(x^1, y^1)$$

for all $(x^1, y^1) \in MP(\Phi/\bar{m})$ *and all* $(x^2, y^2) \in mP(\Phi/\bar{M})$.

Theorem 1 gets

LEMMA 1. $(x^0, y^0) \in SA(\Phi/X \times Y)$ *if and only if* $(x^0, y^0) \in \bar{M} \cap \bar{m}$.

We need also the following well-known lemma:

LEMMA 2. *If* $x^0 \in MP(F/\Delta(x^0))$ *then* $x^0 \in MP(F/D)$, *where* $\Delta(x^0) = \{x \in D : F(x) \leqq F(x^0)\}$.

Now, we can prove:

THEOREM 4. *If* $(x^0, y^0) \in SA(\Phi/X \times Y)$ *and codnition* (A) *holds for every pair of points* $(x^1, y^1) \in \bar{m}$ *and* $(x^2, y^2) \in \bar{M}$, *then*

$$(x^0, y^0) \in mP(\Phi/\bar{M}) \cap MP(\Phi/\bar{m}).$$

Proof. By Lemma 1, $(x^0, y^0) \in \bar{M} \cap \bar{m}$. Suppose that $(x^0, y^0) \notin mP(\Phi/\bar{M})$. Then there exists $(x^2, y^2) \in \bar{M}$ such that $\Phi(x^2, y^2) \leq \Phi(x^0, y^0)$, in contradiction with Theorem 2. If $(x^0, y^0) \notin MP(\Phi/\bar{m})$, then there exists $(x^1, y^1) \in \bar{m}$, such that, $\Phi(x^0, y^0) \leq \Phi(x^1, y^1)$, contradicting again Theorem 2.

THEOREM 5. *If* $(x^1, y^1) \in \bar{m}$, $(x^2, y^2) \in \bar{M}$ *and*

$$\Phi(x^1, y^1) = \Phi(x^2, y^2) = \Phi(x^2, y^1)$$

then $(x^2, y^1) \in SA(\Phi/X \times Y)$.

Proof. Suppose that $(x^2, y^1) \notin SA(\Phi/X \times Y)$, i.e. $(x^2, y^1 \notin \bar{m} \cap \bar{M}$, which implies that $(x^2, y^1) \notin \bar{m}$ or $(x^2, y^1) \notin \bar{M}$. In the first case, there exists $\bar{x} \in X$, such that

$$\Phi(\bar{x}, y^1) \leq \Phi(x^2, y^1) = \Phi(x^1, y^1)$$

so that $(x^1, y^1) \notin \overline{m}$, which is a contradiction. In the second case, i.e. if $(x^2, y^1) \notin \overline{M}$, there exists $\bar{y} \in Y$ such that

$$\Phi(x^2, y^2) = \Phi(x^2, y^1) \leqslant \Phi(x^2, \bar{y}),$$

that is $(x^2, y^2) \notin \overline{M}$, which is absurd.

Consider now a particular case. Let

(17) $$X = R^n,\ Y = \{y \in R^m : y \geqslant 0\} = R^m_+$$

$$G : R^n \to R^m, \text{ and}$$

(18) $$\Phi : R^n \times R^m_+ \to R^p,\ \Phi = (\Phi_1, \ldots, \Phi_p) \text{ be given by}$$

$$\Phi_i(x, y) = F_i(x) + y^T G(x)$$

We will show that in this case condition (A) holds for every pair of points $(x^1, y^1) \in \overline{m}$ and $(x^2, y^2) \in \overline{M}$.

But, let us prove before:

THEOREM 6. *We have* $\overline{M} = \Omega$, *where* Ω *is defined by*

$$\Omega = \{(x, y) \in X \times Y : y^T G(x) = 0 \text{ and } G(x) \leqslant 0\},\ y \in R^m_+.$$

Proof. Suppose that there exists a point $(x^2, y^2) \in \overline{M} \setminus \Omega$. Then
(i) there exists j such that $G_j(x^2) > 0$, or
(ii) $y^T G(x) \neq 0$ and $G(x) \leqslant 0$

In the first case, there exists $\bar{y}_j > y^2_j$. Taking $\bar{y} = (y^2_1, \ldots, \bar{y}_j, \ldots, y^2_m)$ one obtains $\Phi(x^2, \bar{y}) > \Phi(x^2, y^2)$, contradicting the fact that $(x^2, y^2) \in \overline{M}$. In the second case, if $G(x^2) \leqslant 0$, and $y^{2T} G(x^2) \neq 0$, it follows that $y^{2T} G(x^2) < 0$. Putting $\bar{y} = (0, \ldots, 0)$ one obtains $\Phi(x^2, y^2) < \Phi(x^2, \bar{y})$, in contradiction with $(x^2, y^2) \in \overline{M}$.

Suppose now that $(x^2, y^2) \in \Omega$ and $(x^2, y^2) \notin \overline{M}$. Then there exists $\bar{y} \in Y$ such that $\Phi(x^2, y^2) \leqslant \Phi(x^2, \bar{y})$, which means that $0 = y^{2T} G(x^2) < y^T G(x^2)$, which is impossible, since $y^T G(x^2)$ is a sum of negative numbers.

THEOREM 7. *Let* X, Y *and* Φ, *be given by* (15) *and* (16), *respectively. Then condition* (A) *holds for every pair of points* $(x^1, y^1) \in X \times Y$ *and* $(x^2, y^2) \in \overline{M} = \Omega$.

Proof. For $(x^2, y^2) \in \Omega$ *it follows* $G(x^2) \leqslant 0$ and $y^{2T} G(x^2) = 0$, so that $y^{1T} G(x^2) \leqslant y^{2T} G(x^2)$, for all $y^1 \in R^m_+$. But then

$$\Phi(x^2, y^1) \leqslant \Phi(x^2, y^2).$$

REFERENCES

1. Draguşin, C., *Probleme de optimizare cu mai multe funcţii obiectiv*. Teză de doctorat, Bucureşti, 1978.
2. Fuchs, W. D., *Contribuţii la studiul dualităţii în programarea matematică. Probleme de optim vectorial*. Teză de doctorat, Timişoara, 1980.
3. Schönfeld, P., *Some duality Theorems for the non-linear vector maximum problem*, Unternehmensforschung, vol. 14, 1970.
4. Soer, J., Witzgall, C., *Convexity and optimization in finite dimension*, Springer, 1970.

Received 10.V.1983

Universitatea din Braşov
2200 Braşov
România