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SOME PARTICULAR CASES OF DUAL PROBLEMS IN PARETO OPTIMIZATION

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This paper is concerned with the problem of finding the set of all Pareto minimum points of a function $F: R^n \to R^p$ defined on a domain D given by

(1) $$D = \{x \in R^n : G(x) \leqq 0\}$$

where

(2) $$G: R \to R^m.$$

The relation $x \leqq y$ between two elements $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$ in R^m means that $x_i \leqslant y_i$ for $i = 1, \ldots, m$ and $x \leqslant y$ means that $x_i \leqslant y_i$, $i = 1, \ldots, m$ and $x \neq y$.

In the following we shall suppose that the components of the functions F and G, F_i, $i = 1, \ldots, p$, and G_i, $i = 1, \ldots, m$, are convex functions on R^n. It follows that the domain D defined by (1) is convex. Denote by mP(F/D) and by MP(F/D) the set of all Pareto minimum points of F on D, i.e.

(3) $$\mathrm{mP(F/D)} = \{x \in D : \overline{\exists} y \in D, F(y) \leqslant F(x)\},$$

and, respectively, the set of all Pareto maximum points of F on D, i.e.

(4) $$\mathrm{MP(F/D)} = \{x \in D : \overline{\exists} y \in D, F(x) \leqslant F(y)\}$$

Put also

(5) $$M = \{i_1, i_2, \ldots, i_s\} \subset \{1, 2, \ldots, p\}$$

(6) $$N = \{1, 2, \ldots, p\} \setminus M$$

(7) $$\varphi = (\varphi_1, \ldots, \varphi_s), \text{ where } \varphi_j = F_{i_j} \text{ for } i_j \in M, \text{ and}$$

(8) $$D(x^0) = \{x \in D : F_i(x) \leqslant F_i(x^0), i \in N\}, \text{ for } x^0 \in D.$$

Now, we can prove:

THEOREM 1. *The following assertions are equivalent*

(*i*) $x^0 \in \mathrm{mP(F/D)}$

(*ii*) *For every non-void subset M of $\{1, 2, \ldots, p\}$ it follows that* $x^0 \in \mathrm{mP}(\varphi/D(x^0))$, *where φ and $D(x^0)$ are defined, respectively, by* (7) *and* (8).

Proof. (*ii*)⇒(*i*). Follows immediately putting $M = \{1, \ldots, p\}$. In this case $N = \emptyset$ and $D(x^0) = D$.

(*i*) ⇒ (*ii*). Suppose that there exist a non-void subset M of $\{1, \ldots, p\}$ and $x^0 \in \mathrm{mP}(F/D) \setminus \mathrm{mP}(\varphi/D(x^0))$. Then there exist $x^1 \in D(x^0)$ such that $F_i(x^1) \leqslant F_i(x^0)$, for all $i \in M$, and there exists $i_0 \in M$ such that $F_{i_0}(x^1) < F_{i_0}(x^0)$. But x^1 is in $D(x^0)$, i.e. $F_i(x^1) \leqslant F_i(x^0)$ for all $i \in N$, so that $F_i(x^1) \leqslant F_i(x^0)$ for all $i \in \{1, \ldots, p\}$ and $F_{i_0}(x^1) < F_{i_0}(x^0)$, which means that $F(x^1) \leqslant F(x^0)$, implying $x^0 \notin \mathrm{mP}(F/D)$, in contradiction with the choice of x^0.

We say that the system

$$\begin{cases} G(x) \leqq 0 \\ F(x) \leqq F(x^0) \end{cases} \tag{9}$$

verifies *condition A* in x^0 if there exists $i_0 \in \{1, \ldots, p\}$ such that the systems

$$\begin{cases} G(x) \leqq 0 \\ F_i(x) \leqslant F_i(x^0),\ i \in \{1, \ldots, p\} \setminus \{i_0\} \end{cases} \tag{10}$$

and

$$\begin{cases} G(x) \leqq 0 \\ F_{i_0}(x) \leqslant F_{i_0}(x^0) \end{cases} \tag{11}$$

satisfy condition B defined bellow.

A system $G(x) \leqq 0$ verifies *condition B* iff there exists $\bar{x} \in R^n$ such that $G_i(\bar{x}) < 0$ for those $i \in \{1, \ldots, m\}$ for which G_i is nonlinear and $G_i(\bar{x}) \leqslant 0$ if G_i is affine (see Dragomirescu-Maliţa [2], condition 21′ on page 162). Condition B is a regularity condition of Slater type.

THEOREM 2. *Let* $F = (F_1, \ldots, F_p) : R^n \to R^p$ *and* $G = (G_1, \ldots, G_m) : R^n \to R^m$ *be two convex vector functions differentiable on the domain D defined by* (1) *and suppose that system* (9) *verifies condition A on a point* $x^0 \in D$. *Then* $x^0 \in \mathrm{mP}(F/D)$ *if and only if there exist the multiplicators* $\lambda \in R^p$, $\lambda > 0$ *and* $\mu \in R^m$, $\mu \geqq 0$ *such that*

$$\begin{cases} \lambda^T \nabla F(x^0) + \mu^T \nabla G(x^0) = 0 \\ \mu^T G(x^0) = 0 \end{cases} \tag{12}$$

Proof. The sufficiency part of the theorem is known, see. i. Fuchs [4], Th. 6.3.

Necessity. The proof proceeds by induction on p.

For $p = 1$ one obtains the well known Kuhn-Tucker theorem (see e.g. [3], p. 119).

Suppose now that the teorem is true for $p = k - 1$. Without loosing the generality we can suppose that condition A is satisfied in x^0 for $i_0 = k$. By Theorem 1 it follows that $x^0 \in \mathrm{mP}(\varphi/D(x^0))$, where

$$\varphi_i = F_i, \text{ for } i \in M = \{1, \ldots, k-1\}, \text{ and}$$
$$D(x^0) = \{x \in D : F_k(x) \leqslant F_k(x^0)\}.$$

Condition A being satisfied by the system (9) in x^0 it follows that it is satisfied in x^0 by the system

$$\begin{cases} G(x) \leqq 0 \\ F_i(x) \leqslant F_i(x^0),\ i \in M. \end{cases}$$

But, applying then Theorem 2 for the case $p = k - 1$, there exist $\lambda^1 \in R^{k-1}$, $\lambda > 0$ and $\mu^1 \in R^m$, $\mu^1 \geqq 0$, such that

$$\begin{cases} \sum_{i=1}^{k-1} \lambda_i^1 \nabla F_i(x^0) + \mu^{1T} \nabla G(x^0) = 0, \text{ and} \\ \mu^{1T} G(x^0) = 0 \end{cases} \tag{13}$$

Applying now Theorem 1 if follows

$$\min F_k(x) = F_k(x^0), \text{ for } x \text{ in the set } \{x \in R^n : G(x) \leqq 0,\ F_i(x) \leqslant F_i(x^0),\ i \in M\} \text{ where } M = \{1, \ldots, k-1\}. \tag{14}$$

The restriction system

$$\begin{cases} G(x) \leqq 0 \\ F_i(x) \leqslant F_i(x^0),\ i \in M\,; \end{cases}$$

verifies the hypotheses of Kuhn-Tucker theorem and therefore there exist $\lambda^2 \in R^{k-1}$, $\lambda^2 \geqq 0$, and $\mu^2 \in R^m$, $\mu^2 \geqq 0$, such that

$$\begin{cases} \sum_{i=1}^{k-1} \lambda_i^2 \nabla F_i(x^0) + \nabla F_k(x^0) + \mu^{2T} \nabla G(x^0) = 0, \text{ and} \\ \mu^{2T} G(x^0) = 0 \end{cases} \tag{15}$$

By adding (13) and (15) one obtains

$$\begin{cases} \lambda^T \nabla F(x^0) + \mu^T \nabla G(x^0) = 0 \\ \mu^T G(x^0) = 0 \end{cases} \tag{16}$$

where $\lambda_i = \lambda_i^1 + \lambda_i^2 > 0$, for $i \in M$, $\lambda_k = 1 > 0$, $\mu = \mu^1 + \mu^2 \geqq 0$. The theorem is completely proved.

From this theorem one obtains a sufficient condition for a point to be a proper efficient point (for definition see [5]).

COROLLARY 1. *If* $F = (F_i)_{i=1,p}$ *and* $G = (G_j)_{j=1,p}$ *and* F_i, G_j *are convex differentiable functions on the domain D defined by* (1) *and if system* (9) *satisfies condition A in* x^0, *then* x^0 *is a proper efficient point for F on D.*

Proof. By Theorem 2 follows the existence of $\lambda \in R^p$, $\lambda > 0$, such that

$$\min_{x \in D} \left(\sum_{i=1}^{p} \lambda_i F_i(x) \right) = \sum_{i=1}^{p} \lambda_i F_i(x^0) \tag{17}$$

But this relation is a sufficient condition for x^0 to be a proper efficient point (see Geofrion [5]).

As was shown by Tuy [9] condition B is sufficient for the stability of systems (10) and (11). Benson and Morin [1] gave another suffient condition for proper efficiency based also on the stability of a system o inequalities.

By using Theorem 2 we can prove that if F and G are affine then every efficient point is a proper efficient point, a well known property.

COROLLARY 2. *If F_1 is strictly convex and differentiable on the domain D defined by* (1) *and F_i, $i = 2, \ldots, p$, G_i, $i = 1, \ldots, m$ are affine functions, then every Pareto minimum point x^0 of F on D is a proper efficient point, excepting the point x^1 for which $F_1(x') = \min\limits_{x \in D} F_1(x)$.*

Proof. One can see that system (9) verifies condition A in every point $x^0 \neq x^1$ and Corrollary 1 can by applied.

Let

$$\text{(18)} \qquad \begin{cases} \Phi : X \times Y \to R^p \\ X = R^n,\ Y = R^m_+,\ R^m_+ = \{x \in R^m : x \geqq 0\} \end{cases}$$

We say that (x^0, y^0) is a *Pareto saddle point* for Φ on $X \times Y$ if there are no points $x \in X$ and $y \in Y$ such that

$$\Phi(x, y^0) \leqslant \Phi(x^0, y^0) \text{ and } \Phi(x^0, y^0) \leqslant \Phi(x^0, y)$$

We shall denote this situation by writting $(x^0, y^0) \in \mathrm{SA}(\Phi/X \times Y)$

Define now, as in [6], the sets

$$\text{(19)} \qquad \overline{M} = \bigcup_{x \in X} \{(x, y) \in X \times Y : y \in \mathrm{MP}(\Phi(x, .)/Y),\ \text{and}$$

$$\text{(20)} \qquad \overline{m} = \bigcup_{y \in Y} \{(x, y) \in X \times Y : x \in \mathrm{mP}(\Phi(., y)/X).$$

In the following the function Φ will be given by

$$\text{(21)} \qquad \Phi_i(x, y) = F_i(x) + y^T G(x),\ i = 1, \ldots, p.$$

where

$$\text{(22)} \qquad X = R^n,\ Y = R^m_+$$

THEOREM 3. *Let $\Phi = (\Phi_i)_{i=1,p}$ be given by* (21).
(*i*) *If $(x^0, y^0) \in \mathrm{SA}(\Phi/X \times Y)$ then there exists $\lambda \in R^p$, $\lambda \geqq 0$, such that (x^0, y^0) is a saddle point for the scalar function $\lambda^T\Phi$ on $X \times Y$.*
(*ii*) *If there exists $\lambda \in R^p$, $\lambda > 0$, such that (x^0, y^0) is a saddle point for the scalar function $\lambda^T\Phi$, then $(x^0, y^0) \in \mathrm{SA}(\Phi/X \times Y)$.*

Proof. By Theorem 4 in [6], follows $x^0 \in \mathrm{mP}(F/D)$ and then, by Theorem 7.4.1. in [8], one obtains the desired conclusion.
(*ii*) Supposing $(x^0, y^0) \in \mathrm{SA}(\Phi/X \times Y)$, one can arise two situations:
(ii_1) there exists $x \in X$ such that $\Phi(x, y^0) \leqslant \Phi(x^0, y^0)$ implying $\lambda^T\Phi(x, y^0) < \lambda^T\Phi(x^0, y^0)$, a contradiction, or

(ii_2) there exists $y \in Y$ such that $\Phi(x^0, y^0) \leqslant \Phi(x^0, y)$, implying $\lambda^T\Phi(x^0, y^0)$, again in contradiction with the hypotheses that (x^0, y^0) is a saddle point for $\lambda^T\Phi$ on $X \times Y$.

THEOREM 4. *Let $(x^0, y^0) \in \mathrm{mP}(\Phi/\overline{M})$ where $\overline{M}$ is defined by* (19) *and Φ is defined by* (21) *and suppose that the functions F_i, $i = 1, \ldots, p$, G_i, $i = 1, \ldots, m$ are convex and differentiable on the domain D defined by* (1). *If (x^0, y^0) is a proper efficient point then there exists $\overline{y} \in R^m_+$ such that $(x^0, y) \in \mathrm{MP}(\Phi/\overline{m})$, where $\overline{m}$ is defined by* (20).

Proof. In [6] we have proved that

$$\text{(23)} \qquad \overline{M} = \{(x, y) \in R^n \times R^m_+;\ G(x) \leqq 0,\ y^T G(x) = 0\}.$$

Therefore, if (x^0, y^0) is proper efficient then there exists $\lambda \in R^p$, $\lambda > 0$, $\sum\limits_{i=1}^{p} \lambda_i = 1$ such that

$$\min_{(x, y) \in \overline{M}} \lambda^T\Phi(x, y) = \lambda^T\Phi(x^0, y^0)$$

But, by (23) and the definition of Φ it follows that

$$\text{(24)} \qquad \min_{x \in D} \lambda^T F(x) = \lambda^T F(x^0)$$

Taking into account Fritz-John necessary condition (see [3], p. 101) one can find $\overline{y} \in R^m$, $\overline{y} \geqq 0$, such that $(x^0, \overline{y})$ be a saddle point for the function

$$\begin{cases} \lambda^T \Phi(x, y) = \lambda^T F(x) + y^T G(x) \text{ on } R^n \times R^m_+ \text{ and} \\ \overline{y}^T G(x^0) = 0 \end{cases}$$

Therefore, by Theorem 3 (*ii*) it follows $(x^0, \overline{y}) \in \mathrm{SA}(\Phi/X \times Y)$ and by [6] Theorem 4, $(x^0, y)\ \mathrm{MP}(\Phi/\overline{m})$.

Define now the primal and dual Pareto optimum problems:
The primal problem (P): Determine the set $\mathrm{mP}(\Phi/\overline{M})$.
The dual problem (D): Determine the set $\mathrm{MP}(\Phi/\overline{m})$.

Remark that Theorem 4 gives sufficient conditions in order that the existence of a solution of problem (P) imply the existence of a solution of problem (D).

COROLLARY 3. *Suppose that F_1 is strictly convex and differentiable on the domain D defined by* (1), *F_i, $i = 2, \ldots, p$ and G_i, $i = 1, \ldots, m$ are affine functions, $x^0 \in \mathrm{mP}(F/D)$ and $x^0 \neq x^1$, where $\min\limits_{x \in D} F_1(x) = F_1(x^1)$. Then there exists $\overline{y} \in R^m$, $\overline{y} \geqq 0$ such that $(x^0, \overline{y}) \in \mathrm{MP}(\Phi/\overline{m})$.*

Proof. Follows from the Corollary 2 and Theorem 4.

In order to obtain results concerning the reverse connection between problems (D) and (P) we shall study some particular cases in which we know the set $\overline{m}$.

THEOREM 5. *Suppose that the following conditions hold:*
(*i*) Φ *is defined by* (21);
(*ii*) $\overline{m}$ *is defined by* (20);
(*iii*) F_1 *is strictly convex and differentiable on the domain D defined by* (1); *and*
(*iv*) $F_i, i = 2, \ldots, p$ *and* G_i, $i = 1, \ldots, m$ *are affine functions. Then*

(25)
$$\overline{m} = \left\{(x, y) \in R^n \times R^m_+ : \sum_{i=1}^{p} \lambda_i \nabla_x F_i(x) + y^T \nabla_x G(x) = 0, \lambda_i > 0, \sum_{i=1}^{p} \lambda_i = 1\right\}$$

Proof. Denote by ω the right member of equality (25). If $(\overline{x}, \overline{y}) \in \overline{m}$ then, by the definition of $\overline{m}$, $\overline{x} \in \text{mP}(\Phi(., \overline{y})/R^n)$. By Theorem 1

$$\min_{x \in \Delta} (F_1(x) + \overline{y}^T G(x)) = F_1(\overline{x}) + y^T G(\overline{x}),$$

where Δ is the set of solutions of the following system of inequalities:

(26)
$$F_i(x) + \overline{y}^T G(x) \leq F_i(\overline{x}) + \overline{y}^T G(\overline{x}), i = 2, \ldots, p$$

The system (26) satisfies condition B so that, by Kuhn-Tucker theorem, follows the existence of a $\lambda \in R^{p-1}$, $\lambda \geqq 0$, such that

(27)
$$\nabla F_1(x) + y^T \nabla G(x) + \sum_{i=2}^{p} \lambda_i(\nabla F_i(x) + y^T \nabla G(x)) = 0$$

Denoting

$$\overline{\lambda}_1 = 1/\left(1 + \sum_{i=2}^{p} \lambda_i\right) > 0, \ \overline{\lambda}_i = \lambda_i/\left(+ \sum_{i=2}^{p} \lambda_i\right), \ i = 2, \ldots, p,$$

one obtains:

(28)
$$\sum_{i=1}^{p} \overline{\lambda}_i \nabla F_i(x) + y^T \nabla G(x) = 0, \sum_{i=1}^{p} \overline{\lambda}_i = 1, \ \overline{\lambda}_1 > 0$$

so that $\overline{m} \subseteq \omega$.

In order to prove the reverse inclusion observe that for every $\lambda \in R^p$, $\lambda \geqslant 0$, $\lambda_1 > 0$, by the strict convexity of F_1, the set

$$\left\{(x, y) \in R^n \times R^p_+ : \sum_{i=1}^{p} \lambda_i \nabla F_i(x) + y^T \nabla G(x) = 0\right\}$$

is non-void. But then, the function $\sum_{i=1}^{p} \lambda_i F_i$ is also strictly conex, for $\lambda \geqslant 0$, $\lambda_1 > 0$. Therefore, all of the implications from the first part of the proof can be reversed, giving $\omega \subseteq \overline{m}$. The theorem is proved.

THEOREM 6. *Suppose that the following conditions hold:*
(*i*) $(\overline{x}, \overline{y}) \in \text{MP}(\Phi/\overline{M})$;
(*ii*) Φ *is defined by* (19);
(*iii*) F_1 *is strictly convex and differentiable on the domain D defined by* (1);
(*iv*) F_i, $i = 2, \ldots, p$, *and* G_i, $i = 1, \ldots, m$ *are affine functions;*
(*v*) *there exists* $\lambda \in R^p$, $\lambda > 0$, *such that*

(29)
$$\max_{(x, y) \in \overline{m}} \lambda^T \Phi(x, y) = \lambda^T \Phi(\overline{x}, \overline{y}).$$

Then there exists an element $(x^1, y^1) \in R^n \times R^p$, *belonging to* $\text{mP}(\Phi/\overline{M})$.

Proof. Put

(30)
$$m_\lambda = \{(x, y) \in R^n \times R^m_+ : \lambda^T \nabla F(x) + y^T \nabla G(x) = 0$$

In the proof of Theorem 5 it was shown that $\lambda \geqq 0$ and $\lambda_1 > 0$ implies $m_\lambda \neq \Phi$ and $m_\lambda \subset \overline{m}$. Therefore there exists $(x^1, y^1) \in m_\lambda$ such that

$$\max_{(x, y) \in m_\lambda} \lambda^T \Phi(x, y) = \lambda^T \Phi(x^1, y^1).$$

By [3], p. 153, (iii$_3$) it follows

$$\begin{cases} G(x^1) \leqq 0 \\ y^{1T} G(x^1) = 0 \end{cases}$$

By [6], Theorem 6 it follows that $(x^1, y^1) \in \overline{M}$, so that $(x^1, y^1) \in \overline{M} \cap \overline{m}$. But applying [6], Lema 1 one obtains $(x^1, y^1) \in \text{SA}(\Phi/R^n \times R^m_+)$ and so $(x^1, y^1) \in \text{mP}(\Phi/\overline{M})$ [6] Theorem 4.

Knowing the form of the sets $\overline{m}$ and $\overline{M}$ we can prove:

THEOREM 7. *Suppose:*
(*i*) Φ *is defined by* (19);
(*ii*) F_1 *is strictly convex and differentiable on the domain D defined by* (1);
(*iii*) F_i, $i = 2, \ldots, p$, *and* G_i, $i = 1, \ldots, m$ *are affine functions. Then* $x \in \text{mP}(F/D)$ *if and only if x is a solution of the multi-parameter convex programming problem* (P_λ) *given by*

(31)
$$\min_{x \in D} \left(F_1(x) + \sum_{i=2}^{p} \lambda_i F_i(x)\right), \ \lambda_i \geqslant 0, i = 2, \ldots, p.$$

Proof. By Corollary 3, if $\overline{x} \in \text{mP}(F/D)$, and $\overline{x} \neq x^1$, where x^1 is given by

(32)
$$\min_{x \in D} F_1(x) = F_1(x^1).$$

then there exists $\overline{y}$ such that

$$(\overline{x}, \overline{y}) \in \text{SA}(\Phi/R^n \times R^m_+),$$

so that $(x, y) \in \overline{m} \cap \overline{M}$. Therefore

$$(33)\qquad \begin{cases} \sum_{i=1}^{p} \lambda_i \nabla F_i(x) + y^T \nabla G(x) = 0 \\ G(x) \leqq 0 \\ y^T G(x) = 0 \end{cases}$$

One can see that the system (33) has x^1 as solution for $\lambda_i = 0$, $i = 2, \ldots, p$. Consequently, if $\overline{x} \in \text{mP}(F/D)$ then $\overline{x}$ is a solution of (33) which means that it is also a solution of the problem (P_λ). Now suppose that $\overline{x}$ is solution of (33).

By Theorems 6 and 4 in [6], $\overline{x}$ will be an element of the set $\text{mP}(F/D)$. The theorem is proved.

This theorem is important because it makes possible to find all points of the set $\text{mP}(F/D)$ when F_1 is quadratic and positively defined and all of the other functions are affine.

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