MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 15, N° 1, 1986, pp. 49—55

PIECEWISE LINEAR FUNCTIONS IN $\mathbb{R}^3$

IVAN MEŠKO
(Maribor)

Abstract. In this paper an expression of piecewise linear continuous function of three variables using the zero-one variables subject to additional constraints is given.

1. Introduction. Because of the fast development of the integer proggramming and the corresponding computational technics [1] the piecewise linear functions of one [2], [3] and more variables can be useful in mathematical programming [4] and in statistics. By the piecewise linear functions approximation and the linear interpolation for functions of more variables are possible.

Let $f(x, y)$ be a continuous function of two variables and let the values

$$f_{ij} = f(x_i, y_j),\ i = 0, 1, \ldots, m,\ j = 0, 1, \ldots, m$$

$$x_i = a_0 + a_1 + \ldots + a_i,\ i = 0, 1, \ldots, m,\ a_i > 0,\ i = 1, \ldots, m$$

$$y_j = b_0 + b_1 + \ldots + b_j,\ j = 0, 1, \ldots, n,\ b_j > 0,\ j = 1, \ldots, n$$

be known. It can be shown [4] that the function

$$g(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} ((v_{i-1j-1} - v_{i-1j}) f_{i-1j-1} + (f_{ij-1} - f_{i-1j-1}) x_{ij} + \\ + (f_{ij} - f_{i-1j}) x'_{ij} + (f_{i-1j} - f_{i-1j-1}) y_{ij}$$

subject to

$$x = \sum_{i=0}^{m-1} a_i u_i + \sum_{i=1}^{m} \sum_{j=1}^{n} a_i (x_{ij} + x'_{ij})$$

$$y = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} b_j v_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} b_j y_{ij}$$

$$u_i - u_{i+1} = v_{i0},\ i = 0, 1, \ldots, m-1,\ u_0 = 1,\ u_m = 0$$

$$v_{ij} - v_{ij+1} \geqq 0,\ v_{in} = v_{mj} = 0,\ i = 0, 1, \ldots, m-1,\ j = 0, 1, \ldots, n-1$$

$$x_{ij} + x'_{ij} + y_{ij} \leqq 2(v_{i-1j-1} - v_{i-1j}),\ i = 1, \ldots, m,\ j = 1, \ldots, n$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x'_{ij} \leqq t \leqq \sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}+y_{ij}) \leqq 1$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(x_{ij}+x'_{ij}) \leqq 1,$$

where u_i, v_{ij}, t are $0-1$ variables and x_{ij}, x'_{ij}, y_{ij} are non-negative, is piecewise linear and satisfies the condition

$$g(x_i, y_j) = f(x_i, y_j), i = 0, 1, \ldots, m, \; j = 0, 1, \ldots, n.$$

Continuous function of three variables can similarly be expressed.

2. The approximation of the continuous functions in $\mathbb{R}^3$. Let $f(x, y, z)$ be a continuous function of three variables subject to

(1) $$x_0 \leqq x \leqq x_h, \; y_0 \leqq y \leqq y_m, \; z_0 \leqq z \leqq z_n$$

and the values

$$f_{ijk} = f(x_i, y_j, z_k)$$

$$x_i = a_0 + a_1 + \ldots + a_i, \; i = 0, 1, \ldots, h, \; a_i > 0, \; i = 1, \ldots, h$$

$$y_j = b_0 + b_1 + \ldots + b_j, \; j = 0, 1, \ldots, m, \; b_j > 0, \; j = 1, \ldots, m$$

$$z_k = c_0 + c_1 + \ldots + c_k, \; k = 0, 1, \ldots, n, \; c_k > 0, \; k = 1, \ldots, n$$

be given.

Consider the substitution

(2) $$x = \sum_{i=0}^{h-1} a_i u_i + \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(x_{ijk} + x'_{ijk} + x''_{ijk})\, a_i$$

(3) $$y = \sum_{i=0}^{h-1}\sum_{j=0}^{m-1} b_j v_{ij} + \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(y_{ijk} + y'_{ijk})\, b_j$$

(4) $$z = \sum_{i=0}^{h-1}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1} c_k w_{ijk} + \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(z_{ijk} + z'_{ijk})\, c_k$$

subject to

(5) $v_{hj} = v_{im} = 0 \;\; i = 0, 1, \ldots, h, \; j = 0, 1, \ldots, m$

(6) $w_{hjk} = w_{imk} = w_{ijn} = 0, \; i = 0, 1, \ldots, h, \; j = 0, 1, \ldots, m, \; k = 0, 1, \ldots, n$

(7) $u_i - u_{i+1} = v_{i0}, \; i = 0, 1, \ldots, h-1, \; u_0 = 1, \; u_h = 0$

(8) $v_{ij} - v_{ij+1} = w_{ij0}, \; i = 0, 1, \ldots, h-1, \; j = 0, 1, \ldots, m-1$

(9) $w_{ijk-1} - w_{ijk} \geqq 0, \; i = 0, 1, \ldots, h-1, \; j = 0, 1, \ldots, m-1, \;\; k = 1, \ldots, n$

(10) $x_{ijk} + x'_{ijk} + x''_{ijk} + y_{ijk} + y'_{ijk} + z_{ijk} + z'_{ijk} \leqq 3(w_{i-1j-1k-1} - w_{i-1j-1k})$.
$i = 1, \ldots, h, \; j = 1, \ldots, m, \; k = 1, \ldots, n.$

(11) $$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} x'_{ijk} \leqq t_1 \leqq \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(x_{ijk} + y_{ijk} + z_{ijk}) \leqq 1$$

(12) $$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} x''_{ijk} \leqq t_2 \leqq \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(x_{ijk} + x'_{ijk} + z_{ijk}) \leqq 1$$

(13) $$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} y'_{ijk} \leqq t_3 \leqq \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(y_{ijk} + z_{ijk})$$

(14) $$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n} z'_{ijk} \leqq t_4 \leqq \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(x'_{ijk} + x''_{ijk} + y'_{ijk}) \leqq 1$$

(15) $$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(x_{ijk} + x'_{ijk} + x''_{ijk}) \leqq 1$$

(16) $$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(y_{ijk} + y'_{ijk}) \leqq 1$$

(17) $$\sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(z_{ijk} + z'_{ijk}) \leqq 1$$

where u_i, v_{ij}, w_{ijk} and t_i are 0 or 1 and x_{ijk}, x'_{ijk}, x''_{ijk}, y_{ijk}, y'_{ijk}, z_{ijk} and z'_{ijk} are non-negative.

THEOREM. *The function*

$$g(x, y, z) = \sum_{i=1}^{h}\sum_{j=1}^{m}\sum_{k=1}^{n}(f_{i-1j-1k-1}(w_{i-1,j-1k-1} - w_{i-1j-1k})$$

(18) $$+ (f_{ij-1k-1} - f_{i-1j-1k-1})x_{ijk} + (f_{ijk-1} - f_{i-1jk-1})\, x'_{ijk}$$
$$+ (f_{ij-1k} - f_{i-1j-1k})\, x''_{ijk} + (f_{i-1jk-1} - (f_{i-1j-1k-1})\, y_{ijk}$$
$$+ (f_{i-1jk} - f_{i-1j-1k})y'_{ijk} + (f_{i-1j-1k} - f_{i-1j-1k-1})z_{ijk}$$
$$+ (f_{ijk} - f_{ijk-1})z'_{ijk}$$

subject to (2)–(17) *is unique and satisfies the condition*

(19) $g(x_i, y_j, z_k) = f_{ijk}, \; i = 0, 1, \ldots, h, \; j = 0, 1, \ldots, m, \; k = 0, 1, \ldots, n.$

Proof. From (7)–(9) it follows

$$u_i = 1 \text{ if } i < p, \; u_i = 0 \text{ otherwise}, \; 0 < p \leqq h$$

$$v_{p-1j} = 1 \text{ if } j < q, \; v_{ij} = 0 \text{ otherwise}, \; 0 < q \leqq m$$

$$w_{p-1q-1k} = 1 \text{ if } k < r, \; w_{ijk} = 0 \text{ otherwise}, \; 0 < r \leqq n.$$

If $i \neq p$ or $j \neq q$ or $k \neq r$ from (10) it follows

$$x_{ijk} = x'_{ijk} = x''_{ijk} = y_{ijk} = y'_{ijk} = z_{ijk} = z'_{ijk} = 0.$$

Therefore from (2)–(4) and (15)–(17) subject to

$$(20)\qquad x_{i-1} < x < x_i,\ y_{j-1} < y < y_j,\ z_{k-1} < z < z_k$$

it follows $p = i$, $q = j$ and $r = k$.
Considering (2)–(4) we define the parameters

$$d_1 = \frac{x - x_{i-1}}{a_i} + \frac{y - y_{j-1}}{b_j} + \frac{z - z_{k-1}}{c_k} = x_{pqr} + x'_{pqr} + x''_{pqr} + y_{pqr} + y'_{pqr} + z_{pqr} + z'_{pqr}$$

$$d_2 = \frac{y - y_{j-1}}{b_j} + \frac{z - z_{k-1}}{c_k} = y_{pqr} + y'_{pqr} + z_{pqr} + z'_{pqr}$$

$$d_3 = \frac{x - x_{i-1}}{a_i} + \frac{z - z_{k-1}}{c_k} = x_{pqr} + x'_{pqr} + x''_{pqr} + z_{pqr} + z'_{pqr}.$$

From (11)–(14) and (20) it follows

$$(21)\qquad d_1 \leqq 1 \Rightarrow x'_{pqr} = x''_{pqr} = y'_{pqr} = z'_{pqr} = 0$$

$$(22)\qquad d_1 \geqq 1 \Rightarrow x_{pqr} + y_{pqr} + z_{pqr} = 1$$

$$(23)\qquad d_1 \leqq 2 \Rightarrow z'_{pqr} = 0$$

$$(24)\qquad d_2 \leqq 1 \Rightarrow y'_{pqr} = 0 \text{ and } z'_{pqr} = 0$$

$$(25)\qquad d_2 \geqq 1 \Rightarrow x_{pqr} = 0 \text{ and } y_{pqr} + z_{pqr} = 1$$

$$(26)\qquad d_3 \leqq 1 \Rightarrow x''_{pqr} = 0 \text{ and } z'_{pqr} = 0$$

$$(27)\qquad d_3 \geqq 1 \Rightarrow x_{pqr} + y_{pqr} + z_{pqr} = 1 \text{ and } x_{pqr} + x'_{pqr} + z_{pqr} = 1$$

$$(28)\qquad d_1 \geqq 2 \Rightarrow d_2 \geqq 1,\ d_3 \geqq 1 \text{ and } x'_{pqr} + x''_{pqr} + y'_{pqr} = 1.$$

If $d_1 = 1$ (21) and (22) are equivalent, if $d_1 = 2$ (23) and (28) are equivalent, if $d_2 = 1$ (24) and (25) are equivalent and if $d_3 = 1$ (26) and (27) are equivalent. Therefore x_{ijk}, x'_{ijk}, x''_{ijk}, y_{ijk}, y'_{ijk}, z_{ijk} and z'_{ijk} are independent from the choice of the conditions (21)–(28) if there more feasible choices exist.
We have six diferent cases (see figure):

$S_1(A, B, D, E) : d_1 \leqq 1$

$S_2(B, C, D, E) : d_1 \geqq 1,\ d_2 \leqq 1$ and $d_3 \leqq 1$

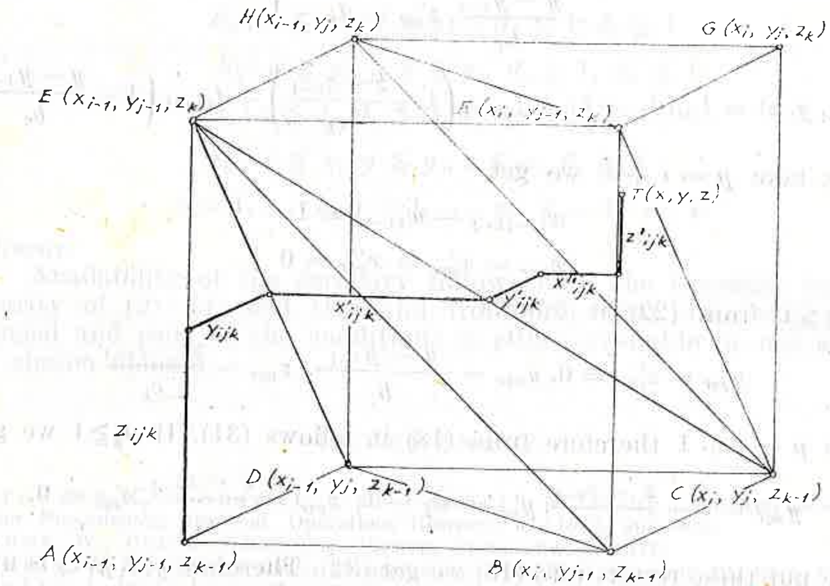
$S_3(B, C, E, F) : d_2 \leqq 1$ and $d_3 \geqq 1$

$S_4(C, D, E, H) : d_2 \geqq 1$ and $d_3 \leqq 1$

$S_5(C, E, F, H) : d_1 \leqq 2,\ d_2 \geqq 1$ and $d_3 \geqq 1$

$S_6(C, F, G, H) : d_1 \geqq 2.$

In each of the six cases by (2)–(4) and (21)–(28) the variables x_{pqr}, x'_{pqr}, x''_{pqr}, y_{pqr}, y'_{pqr}, z_{pqr} and z'_{pqr} are unique determined and (15)–(17) are satisfied too. Since p, q and r are unique determined, $g(x, y, z)$ is unique subject to (20).



Now we take instead of (20)

$$(29)\qquad x = x_i,\ y_{j-1} < y < y_j,\ z_{k-1} < z < z_k.$$

The implications (21)–(28) hold also in this case. If $i < h$ from (2)–(4) and (15)–(17) it follows $p = i$ or $i + 1$, $q = j$ and $r = k$. If we take $p = i$, we get $u_i = 0$ and from (2) it follows

$$(30)\qquad x_{pqr} + x'_{pqr} + x''_{pqr} = 1 \Rightarrow d_1 > \text{ and } d_3 > 1.$$

If $d_2 \leqq 1$ from (2)–(4), (22), (24) and (27) it follows

$$y'_{pqr} = z'_{pqr} = 0,\quad x_{pqr} = 1 - d_2,\ x'_{pqr} = y_{pqr} = \frac{y - y_{j-1}}{b_j},$$

$$x''_{pqr} = z_{pqr} = \frac{z - z_{k-1}}{c_k}.$$

If we put these results and $w_{i-1j-1k-1} - w_{i-1j-1k} = 1$ into (18), we get

$$(31)\qquad g(x_i, y, z) = f_{ij-1k-1}(1 - d_2) + f_{ijk-1}\frac{y - y_{j-1}}{b_j} + f_{ij-1k}\frac{z - z_{k-1}}{c_k}$$

If $d_2 \geqq 1$ it follows $d_1 \geqq 2$ and from (3)–(4), (28), (30) and (18) it follows

$$x_{pqr} = y'_{rqr} = 0,\ x'_{pqr} = y_{pqr} = \frac{y - y_{j-1}}{b_j},\ x''_{pqr} = z_{pqr} = 1 -$$

(32) $$- \frac{y - y_{j-1}}{b_j},\ z'_{pqr} = d_2 - 1$$

$$g(x_i, y, z) = f_{ijk}(d_2 - 1) + f_{ijk-1}\left(1 - \frac{z - z_{k-1}}{c_k}\right) + f_{ij-1k}\left(1 - \frac{y - y_{j-1}}{b_j}\right)$$

If we take $p = i + 1$ we get

$$w_{ij-1k-1} - w_{ij-1k} = 1$$

$$x_{pqr} = x'_{pqr} = x''_{pqr} = 0$$

If $d_2 \leqq 1$ from (22) it follows

$$y'_{pqr} = z'_{pqr} = 0,\ y_{pqr} = \frac{y - y_{j-1}}{b_j},\ z_{pqr} = \frac{z - z_{k-1}}{c_k}.$$

Since $p = i + 1$ therefore from (18) it follows (31). If $d_2 \geqq 1$ we get

$$y_{pqr} = 1 - \frac{z - z_{k-1}}{c_k},\ y'_{pqr} = d_2 - 1,\ z_{pqr} = \frac{z - z_{k-1}}{c_k},\ z'_{pqr} = 0.$$

If we put these results into (18) we get (32). Therefore $g(x, y, z)$ is unique subject to (29).

Similarly it can be proved that the function (18) is unique in other cases. In the case $x = x_i, y = y_j, z = z_k, 0 < i < h, 0 < j < m, 0 < k < < n$ we get

$$p = i \text{ or } i + 1,\ q = j \text{ or } j + 1,\ r = k \text{ or } k + 1.$$

If we take $p = i + 1$, $q = j + 1$ and $r = k + 1$, we get

$$x_{pqr} = x'_{pqr} = x''_{pqr} = y_{pqr} = y'_{pqr} = z_{pqr} = z'_{pqr} = 0$$

$$w_{ijk} - w_{ijk+1} = 1.$$

Therefore from (18) it follows

$$g(x_i, y_j, z_k) = f_{ijk}$$

and $g(x, y, z)$ satisfies condition (19) subject to $i < h$, $j < m$, $k < n$. For $x = x_h$, $y = y_m$, $z = z_n$ considering (2)–(9) we get

$$p = h,\ q = m,\ r = n,\ d_1 = 3,\ w_{h-1m-1n-1} = 1.$$

From (28) and (18) if follows

$$x'_{hmn} = y_{hmn} = z'_{hmn} = 1,\ x_{hmn} = x''_{hmn} = y'_{hmn} = z_{hmn} = 0$$

$$g(x_h, y_m, z_n) = f_{hmn}.$$

Therefore $g(x, y, z)$ satisfies condition (19).

COROLLARY. *The function* $g(x, y, z)$ *is continuous subject to* (1) *and in the domains*

$$S_1 : x \geqq x_{i-1}, y \geqq y_{j-1}, z \geqq z_{k-1}, d_1 \leqq 1$$
$$S_2 : z \geqq z_{k-1}, d_1 \geqq 1, d_2 \leqq 1, d_3 \leqq 1$$
$$S_3 : x \leqq x_i,\ y \geqq y_{j-1}, d_2 \leqq 1, d_3 \geqq 1$$
$$S_4 : x \geqq x_{i-1},\ y \leqq y_j,\ d_2 \geqq 1,\ d_3 \leqq 1$$
$$S_5 : z \leqq z_k, d_1 \leqq 2,\ d_2 \geqq 1,\ d_3 \geqq 1$$
$$S_6 : x \leqq x_i, y \leqq y_j, z \leqq z_k, d_1 \geqq 2$$
$$i = 1, \ldots, h,\ j = 1, \ldots, m,\ k = 1, \ldots, n$$

is linear.

Availability of the corollary follows from the theorem, from the linearity of (2)–(4) and (18) and from (21)–(28) since w_{ijk} are not changed and each of the conditions is either available or not all over the chosen domain.

REFERENCES

1. Crowder, H., Johnson E. L. and Padberg, M., *Solving Large-Scale Zero-One Linear Programming Problems*. Operations Research **31**, 1983, 803–834.
2. Dück, W., *Diskrete Optimierung*. Vieweg, Braunschweig, 1977.
3. Healy, W. C., *Multiple Choice Programming*. Operations Research **12**, 1964, 122–138.
4. Meško, I., *Mixed Integer Model for Optimization of Business Processes* (croate). SYM-OP-IS'85 (Yugoslav symposium of operations research), Herceg Novi 1985, 51–58.

Received 23.X.1985

Professor Ivan Meško
Visoka ekonomsko-komercialna šola
YU 62000 Maribor, Razlagova 14