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NEW ESTIMATES OF THE DEGREE OF THE COMONOTONE INTERPOLATING POLYNOMIALS

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1. The aim of this paper is to give a method for the estimate of the degree of the comonotone interpolating polynomials, based on the use of an arbitrary sequence of approximation operators for which the order of approximation is known. The paper is an addition at our previous paper [5], to which we shall refer.

Let there be given $m \in \mathbb{N}$, $m \geq 1$, the points

$$(x) \qquad 0 = x_0 < x_1 < \ldots < x_m = 1$$

and the real numbers

$$(y) \qquad 0 = y_0, y_1, \ldots, y_m$$

satisfying the condition $y_i - y_{i-1} \neq 0$, $i = 1, 2, \ldots, m$.

It is well known [8], [9] that, under this condition, there is at least one polynomial P satisfying the following two conditions :

$$(1) \qquad P(x_i) = y_i, \quad i = 0, 1, \ldots, m\,;$$

$$(2) \qquad P'(x)(y_i - y_{i-1}) \geq 0, \quad x \in [x_{i-1}, x_i], \quad i = 1, 2, \ldots, m.$$

Such a polynomial is named comonotone interpolating polynomial.

Estimates of the degree of the comonotone interpolating polynomials have been established in [4], [5] and for some particular cases, in [1], [2], [3]. In what follows we shall describe a simple method which permits us to obtain such estimates. This method points out a relation between the degree of the comonotone interpolating polynomials and the order of approximation by polynomials of the continuous real functions.

2. We denote by C [0, 1], the space of all the real functions, continuous on the interval [0, 1], endowed with the uniform norm ; by $\mathscr{P}_n$, its subspace of all the polynomials of degree at most equal to n and by $\omega(f; \cdot)$, the modulus of continuity of the function $f \in C[0, 1]$.

Let us consider a sequence (L_n) of operators $L_n : C[0, 1] \to \mathscr{P}_n$, $n = 1, 2, \ldots,$ satisfying the following conditions:

(3) $$L_n f \geqslant 0, \text{ for any } f \geqslant 0;$$

(4) $$L_n(\lambda f) = \lambda L_n f, \text{ for every } f \in C[0, 1] \text{ and } \lambda > 0;$$

(5) $$\|L_n f - f\| \leqslant K\omega(f; n^{-r}), \text{ for any } f \in C[0, 1],$$

where K and r are two positive constants not depending on f and n.

We shall use the notations:

$$\alpha = \max_{1 \leqslant i \leqslant m} |y_i - y_{i-1}|,\ \beta = \min_{1 \leqslant i \leqslant m} |y_i - y_{i-1}|,\ \gamma = \min_{1 \leqslant i \leqslant m} (x_i - x_{i-1}).$$

Our main result is the following:

THEOREM. *There is at least one polynomial P such that conditions (1), (2) be fulfilled and that degree $P \leqslant n + m$, where n is the smallest integer satisfying the following inequalities:*

(6) $$n > (1/\gamma)^{2/r}$$

(7) $$n > C((\alpha + \beta)/\beta)^{2/r}$$

(*C being a positive constant depending only on K, r and on the points (x)*)

Proof. The symbols $\mathscr{D}$, F, A, $\bar{A}$, λ_j, c_j, σ_j, a_j, $\bar{a}_j$ $(j = 1, 2, \ldots, m)$ that will be used in the sequel, have the meaning from [5].

To each positive number $\varepsilon < \gamma$ we associate the functions $s_j^\varepsilon \in C[0, 1]$, $j = 1, 2, \ldots, m$:

(8) $$s_j^\varepsilon(x) = \begin{cases} 0 & x \notin [x_{j-1} - \varepsilon, x_j + \varepsilon] \\ (x - x_{j-1} + \varepsilon)(1/\varepsilon), & x \in [x_{j-1} - \varepsilon, x_{j-1}] \\ 1 & , x \in [x_{j-1}, x_j] \\ (x - x_j - \varepsilon)(-1/\varepsilon), & x \in [x_j, x_j + \varepsilon] \end{cases}$$

For these functions one has $\omega(s_j^\varepsilon; \delta) \leqslant (1/\varepsilon)\delta$, $j = 1, 2, \ldots, m$. With their aid we approximate the vectors λ_j, $j = 1, 2, \ldots m$ (see [5]) by the vectors:

(9) $$\bar{\lambda}_j = F(c_j(L_n s_j^\varepsilon)Q),\ j = 1, 2, \ldots, m,$$

where $Q \in \mathscr{D}$ is a fixed polynomial of a degree at most equal to $m - 1$.

We shall determine n and $\varepsilon = \varepsilon(n)$ such that for these vectors $\bar{\lambda}_j$, $j = 1, 2, \ldots, m$, condition [5, (13)] be fulfilled.

The elements of the matrix $A + \bar{A} = (\alpha_{ij})_{1 \leqslant i,\, j \leqslant m}$, are

(10) $$\alpha_{ij} = c_j \int_0^{x_i} L_n s_j^\varepsilon(t) Q(t) dt, \quad 1 \leqslant i, j \leqslant m$$

and those of the matrix $A^{-1} = (\sigma_{ij})_{1 \leqslant i,\, j \leqslant m}$, are

$$\sigma_{ij} = \begin{cases} 0, & \text{if } i \neq j,\ i \neq j + 1 \\ \sigma_i, & \text{if } i = j \\ -\sigma_i, & \text{if } i = j + 1 \end{cases}$$

Hence, we deduce that the elements of the matrix $I + A^{-1}\bar{A} = (\beta_{ij})_{1 \leqslant i,\, j \leqslant m}$, are:

(11) $$\beta_{ij} = c_j \int_{x_{i-1}}^{x_i} L_n s_j^\varepsilon(t) |Q(t)| dt, \quad 1 \leqslant i, j \leqslant m.$$

We shall give estimates for the elements of the matrix $A^{-1}\bar{A}$. We have

(12) $$|\beta_{ii} - 1| = \left|c_i \int_{x_{i-1}}^{x_i} L_n s_i^\varepsilon(t) |Q(t)| dt - c_i \int_{x_{i-1}}^{x_i} |Q(t)| dt\right| \leqslant \|L_n s_i^\varepsilon - s_i^\varepsilon\| \leqslant$$
$$\leqslant K\omega(s_i^\varepsilon; n^{-r}) \leqslant K(1/\varepsilon)n^{-r},\ i = 1, 2, \ldots, m.$$

If $i < j - 1$ or $i > j + 1$, then:

(13) $$\beta_{ij} = c_j \int_{x_{i-1}}^{x_i} L_n s_j^\varepsilon(t) |Q(t)| dt \leqslant (c_j/c_i)K(1/\varepsilon)n^{-r},$$

while for $i = j + 1$, we have:

(14) $$\beta_{j+1,j} = c_j \int_{x_j}^{x_j+\varepsilon} L_n s_j^\varepsilon(t) |Q(t)| dt + c_j \int_{x_j+\varepsilon}^{x_{j+1}} L_n s_j^\varepsilon(t) |Q(t)| dt \leqslant$$
$$\leqslant c_j(1 + K(1/\varepsilon)n^{-r})\|Q\|\varepsilon + (c_j/c_{j+1})K(1/\varepsilon)n^{-r}.$$

Similarly

(15) $$\beta_{j-1,j} \leqslant c_j(1 + K(1/\varepsilon)n^{-r})\|Q\|\varepsilon + (c_j/c_{j-1})K(1/\varepsilon)n^{-r}.$$

Now if we put $\varepsilon = n^{-r/2}$ and we consider the constant

$$C' = \max\{K, (c_j/c_i)K, 2c_k\|Q\| + (c_k/c_{k+1})K, 2c_{k+1}\|Q\| + (c_{k+1}/c_k)K;$$
$$i \neq j,\ i \neq j + 1,\ i \neq j - 1,\ i, j = 1, 2, \ldots, m,\ k = 1, 2, \ldots, m - 1\},$$

depending only on K and on the points (x), then by using (12), (13), (14), (15), we see that the condition [5, (13)] is satisfied if the inequalities:

(16) $$Kn^{-r/2} \leqslant 1;$$

(17) $$C'mn^{-r/2} < \beta/(\alpha + \beta),$$

are fulfilled. From (17) we infer that $n > (C'm)^{2/r}((\alpha + \beta)/\beta)^{2/r}$. This inequality is equivalent to relation (7), with $C = (C'm)^{2/r}$ and since $C' \geqslant K$, we see that (17) assures (16).

Now the proof is finished if we observe that a required comonotone interpolating polynomial is the following:

$$P(x) = \sum_{j=1}^{m} (a_j + \tilde{a}_j)c_j \int_0^x L_n s_j^\varepsilon(t)Q(t)dt \quad (\varepsilon = n^{-r/2}). \tag{18}$$

Remarks. 1°. If we denote by $\tilde{L}_n$ the operator which assignes to each function $f \in C[0, 1]$, its best approximation polynomial $\tilde{L}_n f \in \mathscr{P}_n$ then, according to a classical result of D. Jackson (see [7]), we have $\|\tilde{L}_n f - f\| \leqslant (K/2)\omega(f; n^{-1})$ and consequently, for the operator L_n: $C[0, 1] \to \mathscr{P}_n$ defined by $L_n f = \tilde{L}_n f + (K/2)\omega(f; n^{-1})$ $(f \in C[0, 1])$, conditions (3)—(5) with $r = 1$, are fulfilled. Therefore, there is at least one polynomial P satisfying (1), (2) and degree $P \leqslant n + m$, where n is the smallest integer such that $n > (1/\gamma)^2$ and $n > C((\alpha + \beta)/\beta)^2$, C being a determined positive constant depending only on K and on the points (x).

2°. If we are interested in the effective construction, based on formula (18), of a comonotone interpolating polynomial, we may use other sequences (L_n) of operators by which the images of the functions s_j^ε, can be more easily determined. For instance, we may use the sequence (B_n) of the Bernstein's operators; in this case $K = 5/4$ and $r = 1/2$.

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