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VORONOVSKAJA-TYPE THEOREMS FOR A CERTAIN NON-POSITIVE LINEAR OPERATOR

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This paper deals with some Voronovskaja-type theorems for a certain non-positive linear operator in two variables, which is a pseudopolynomial operator in the sense of Marchaud, considered by I. Badea in [2].

1. Introduction. There is a wide variety of both positive and non-positive linear methods in the approximation of multivariate functions.

One of these methods yields the so-called pseudopolynomial operators. Their construction is also based on univariate mappings, but the resulting operator is not necessarily positive, even if the underlying univariate ones are so.

In this paper we shall consider an example of a pseudo-polynomial operator which is based upon the Bernstein operator. For each natural n (here a natural number is a positive integer) and real-valued function $f(x,y)$ defined on $[0,1]^2 = [0,1] \times [0,1]$, the sequence $\{P_n f\}$ is defined by the sum

$$(1.1)\qquad P_n(f;\ x,\ y) = \frac{1}{2}\sum_{i=0}^{n}\left\{f\left(x,\frac{i}{n}\right)+f\left(\frac{i}{n},\ y\right)-f\left(\frac{i}{n},\frac{i}{n}\right)\right\}\{p_{n,i}(x)+p_{n,i}(y)\},$$

where

$$(1.2)\qquad p_{n,i}(x) = \binom{n}{i} x^i(1-x)^{n-i},\ 0 \leqslant i \leqslant n,\ 0 \leqslant x \leqslant 1.$$

For every $f \in C([0,1]^2)$ (the linear space of real-valued continuous functions defined on $[0,1]^2$), the sequence $\{P_n f\}$, $n \geqslant 1$, converges uniformly to f on $[0,1]^2$ (see Badea [2] and Badea and Oprea [3]).

Quantitative versions for these theorems were given by Badea [2] (involving the first modulus of continuity) and recently by Gonska [7] (involving the least concave majorant of the first modulus of continuity).

H.H. Gonska [7] also showed that P_n is not a positive operator. Indeed, for the function $f_0 \geqslant 0$ given for $0 \leqslant x,\ y \leqslant 1$ by

$$f_0(x,\ y) = \begin{cases}(1-\alpha) \text{ if } x+y = /2n,\ 0 \leqslant \alpha \leqslant 1\\ 0 \text{ elsewhere in } [0,1]^2\end{cases}$$

we have $P_n(f;\ 1/2n,\ 1/2n) = -(1-1/2n)^n < 0$.

The aim of the present paper is to prove some Voronovskaja-type theorems for the operator P_n. We shall prove Voronovskaja-type theorems with traditional flavour and, in the third section, we shall prove also a Voronovskaja type theorem for functions which are bidimensionally derivable, a class of functions firstly considered by Bögel [4], [5] (see also [6] for a more recent reference).

2. Some Voronovskaja-type theorems. In the remainder of this paper, we denote by $C^n[0,1]$ the linear space of real-valued functions $f(x)$ defined on $[0,1]$, n-th order derivable, such that $f^{(n)}(x)$ is continuous on $[0,1]$. Also $C^n([0,1]^2)$ is the linear space of real-valued functions $f(x,y)$ defined on $[0,1]^2$ with all partial derivatives $\frac{\partial^{i+k} f}{\partial x^i \partial y^k}(x, y)$, $i, k \geq 0$, $i + k \leq n$ continuous on $[0,1]^2$.

In our theorems from this paragraph we shall use two well-known theorems stated as

LEMMA 1 (*Voronovskaja's theorem*). *If* $g \in C^2([0,1])$ *we have*:

$$\lim_{n\to\infty} n[B_n(g\,;t) - g(t)] = (1/2)\ t(1 - t)g''(t) \tag{2.1}$$

LEMMA 2 (*Sikkema's theorem*). *If* $g \in C^4([0,4])$ *we have*:

$$\lim_{n\to\infty} n\left\{n[B_n(g;\ t) - g(t)] - \frac{t(1-t)}{2} g''(t)\right\} = \tag{2.2}$$

$$= \frac{t(1-t)(1-2t)}{6} g'''(t) + \frac{t^2(1-t)^2}{8} g^{\mathrm{IV}}(t).$$

For a proof of Lemma 1, see Voronovskaja [11] and for Lemma 2 see Sikkema [10].

Then we have

THEOREM 1. *Let* f *be an element of the space* $C^2([0,1]^2)$. *Then*

$$\lim_{n\to\infty} n[P_n(f;\ x,\ y) - f(x,y)] = \frac{x(1-x)}{4}\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{y(1-y)}{4}\frac{\partial^2 f}{\partial y^2}(x, y).$$

Proof. Since $P_n(f;\ x,\ y) = \frac{1}{2}\sum_{i=0}^{n}\left\{f\left(x, \frac{i}{n}\right) + f\left(\frac{i}{n},\ y\right) - f\left(\frac{i}{n}, \frac{i}{n}\right)\right\}\cdot\{p_{n,i}(x) + p_{n,i}(y)\}$, we can write

$$P_n(f;\ x,\ y) = (1/2)B_n(f(x,.);\ x) + (1/2)B_n(f(.,y);\ x) -$$
$$-(1/2)B_n(f(.,.);x) + (1/2)B_n(f(x,.);y) +$$
$$+(1/2)B_n(f(.,y);\ y) - (1/2)B_n(f(.,.);\ y)$$

where $B(f;\ x) = \sum_{i=0}^{n} f(i/n)p_{n,i}(x)$ is the Bernstein operator.

Using now the equality (2.1) from Lemma 1 we have

$$P_n(f;\ x,\ y) - f(x,\ y) = (1/2)[B_n(f(x,.);\ x) - f(x,\ x)] +$$

$$+(1/2)[B_n(f(.,y);x) - f(x,\ y)] - (1/2)\ [B_n(f(.,.);\ x) - f(x,x)] +$$
$$+(1/2)\ [B_n(f(x\,.,);\ y) - f(x,\ y)] + (1/2)\ [B_n(f(.,y);\ y) - f(y,\ y)] -$$
$$-(1/2)\ [B_n(f(.,.);\ y) - f(y,\ y)] =$$
$$= \frac{x(1-x)}{4n}\frac{\partial^2 f}{\partial x^2}(x,\ y) + \frac{y(1-y)}{4n}\frac{\partial^2 f}{\partial y^2}(x,\ y) + \varepsilon_n(x,\ y),$$

where $\lim_{n\to\infty} n\varepsilon_n(x,\ y) = 0$ uniformly in respect with x and y.

Hence the proof of Theorem 1 is complete.

Another result is the following

THEOREM 2. *Let* f *be an element of the space* $C^4([0,1]^2)$. *Then we have*

$$\lim_{n\to\infty} n\left\{n[P_n(f;\ x,\ y) - f(x,\ y)] - \frac{x(1-x)}{4}\frac{\partial^2 f}{\partial x^2}(x, y) - \frac{y(1-y)}{4}\cdot\frac{\partial^2 f}{\partial y^2}(x,\ y)\right\} = \frac{x(1-x)(1-2x)}{12}\frac{\partial^3 f}{\partial x^3}(x,\ y) + \frac{y(1-y)(1-2y)}{12}$$

$$\frac{\partial^3 f}{\partial y^3}(x,y) + \frac{x^2(1-x)^2}{16}\frac{\partial^4 f}{\partial x^4}(x,\ y) + \frac{y^2(1-y)^2}{16}\frac{\partial^4 f}{\partial y^4}(x,\ y).$$

Proof. Using the decomposition of $P_n(f;\ x,\ y)$ with the help of Bernstein polynomials from the proof of Theorem 1 and the equality (2.2) of Lemma we can write the desired relation.

Hence the proof of Theorem 2 is complete.

3. A Voronovskaja-type theorem for bidimensionally derivable functions. In 1934 Karl Bögel [4], [5], [6] introduced the notions of bidimensionally continuous and derivable functions, as natural generalizations of the usual notions from the univariate case. We recall here these definitions.

We say that the real-valued function $f(x,\ y)$ defined on $[0,1]^2$ is *bidimensionally continuous* (B-continuous briefly) in the point $(s,\ t) \in [0,1]^2$ if $\lim_{\substack{x\to s\\ y\to t}} |f(x,\ y) - f(x,\ t) - f(s,\ y) + f(s,\ t)| = 0$ and B-continuous at $A \subset [0,1]^2$ if it is B-continuous in every point of A.

Function $f(x,\ y)$ is *bidimensionally derivable* (B-derivable briefly) in the point $(s,\ t) \in [0,1]^2$ if there is a real number denoted as $f'(s, t)$, which has the following property

$$\lim_{\substack{x\to s\\ y\to t\\ (x-s)(y-t)\neq 0}} \left|\frac{f(x,\ y) - f(x,\ t) - f(x,\ y) + f(s,\ t)}{(x-s)(y-t)} - f'(s,\ t)\right| = 0$$

In the proof of our main result of this paragraph we need the following

LEMMA 3. *If the real-valued function* $f(x,\ y)$ *defined on* $[0,1]^2$ *is* B-*derivable in* $(x_0,\ y_0)$ *then there is a real-valued function* $\varphi(x,\ y)$, *defined*

on $[0,1]^2$, *continuous in the point* (x_0, y_0) *and which vanishes in the points of the set*
$C_0 := \{(x, y) \in [0, 1]^2 : (x - x_0)(y - y_0) = 0\}$ *such that for all* $(x, y) \in [0,1]^2$ *we have*

$$f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0) = \tag{3.0}$$
$$= \{f'(x_0, y_0) + \varphi(s, y)\}(x - x_0)(y - y_0).$$

Proof. We take

$$\varphi(x, y) = \begin{cases} 0, & \text{if } (x, y) \in C_0 \\ \dfrac{f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0)}{(x - x_0)(y - y_0)} - f'(x_0, y_0), & \text{elsewhere in } [0,1]^2 \end{cases}$$

Now we are ready to prove the following

THEOREM 3. *If the real-valued function* $f(x, y)$ *defined on* $[0,1]^2$ *is B-continuous at the whole unit square and B-derivable in* (u,v) *then we have*

$$f(u, v) - P_n(f; u, v) =$$
$$= \frac{1}{2n} f'(u, v)\{u(1 - u) + v(1 - v)\} + o\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{1}{\sqrt{n}}\right) + O(\sqrt{n})$$

As usual, in the above equality o and O are the well-known symbols of Landau.

The appearance of Landau's symbol O in Theorem 3 is justfied by the following example of a B-derivable real-valued function $f(x, y)$ at $[0,1]^2$ for which the operator $P_n(f; x,y)$ does not converge uniformly to $f(x,y)$ on $[0,1]^2$.

Example. If we denote by B the boundary of the unit square $[0,1]^2$ let us consider the following function

$$f_0(x, y) = \begin{cases} 0 & \text{if } (x, y) \in B \\ xy & \text{elsewhere in } [0,1]^2 \end{cases}$$

It is easy to see that $f_0(x, y)$ is B-derivable at $[0,1]^2$ and $f_0'(x, y) = 1$ for every point (x, y) from $[0,1]^2$, but $P_n(f_0; 0, 0)$, $n = 1,2,\ldots$, does not converge to $f_0(0,0) = 0$. Indeed we have $P_n(f_0; 0,0) = -(n + 1)(2n + 1)/6n$, because $p_{n,i}(0) = 1$.

This example extends an example of Badea and Oprea [3] who give a B-continuous function which is not uniformly approximated by the operator P_n.

Proof of Theorem 3. Applying the above equality (3.0) from Lemma 3, we can write

$$f(u, v) - f\left(u, \frac{i}{n}\right) - f\left(\frac{i}{n}, v\right) + f\left(\frac{i}{n}, \frac{i}{n}\right) = \tag{3.1}$$
$$\left\{f'(u, v) + \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\right\}\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right)$$

and

$$\lim_{\substack{x \to u \\ y \to v}} \varphi(x, y) = 0. \tag{3.2}$$

Multiplying equality (3.1) with $(1/2)\{p_{n,i}(x) + p_{n,i}(y)\}$ and adding these equalities after the variable i we get, keeping in mind that $\sum\limits_{i=0}^{n} p_{n,i}(t) = 1$, the following relation

$$f(u, v) - P_n(f; u, v) = \frac{1}{2} f'(u, v) S_n(u, v) + \sigma_n(u, v) \tag{3.3}$$

where

$$S_n(u, v) = \frac{1}{2} \sum_{i=0}^{n} \left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right)\{p_{n,i}(u) + p_{n,i}(v)\} \tag{3.4}$$

and

$$\sigma_n(u, v) = \frac{1}{2} \sum_{i=0}^{n} \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right)\{p_{n,i}(u) + p_{n,i}(v)\} \tag{3.5}$$

Firstly, we shall calculate the sum $S_n(u, v)$ given by (3.4); we have

$$\sum_{i=0}^{n} \left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right) p_{n,i}(u) =$$
$$= \frac{1}{n^2} \sum_{i=0}^{n} i^2 p_{n,i}(u) - \frac{1}{n}(u + v) \sum_{i=0}^{n} i p_{n,i}(u) + uv$$

From the theory of the Bernstein polynomials it is known that

$$\sum_{i=0}^{n} i p_{n,i}(t) = nt, \quad \sum_{i=0}^{n} i^2 p_{n,i}(t) = nt + n(n - 1)t^2$$

Thus we have

$$\sum_{i=0}^{n} \left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right) p_{n,i}(u) = \frac{1}{n} u(1 - u) \tag{3.6}$$

In a similar way, we deduced that

$$\sum_{i=0}^{n} \left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right) p_{n,i}(v) = \frac{1}{n} v(1 - v) \tag{3.7}$$

Adding the last two equalities, we get

$$S_n(u, v) = \frac{1}{n}\{u(1 - u) + v(1 - v)\} \tag{3.8}$$

Now we estimate the expression $\sigma_n(u, v)$. Let us consider the following sets :

$$A = \left\{ i : \left| u - \frac{i}{n} \right| < n^{-\frac{1}{4}} \right\} \subset \{0, 1, \ldots, n\}$$

$$B = \left\{ i : \left| v - \frac{i}{n} \right| < n^{-\frac{1}{4}} \right\} \subset \{0, 1, \ldots, n\}$$

Using these notations, we can write :

$$\sigma_n(u, v) = \sum_{j=1}^{6} \sigma_n^{(j)}(u, v) \tag{3.9}$$

where the sums $\sigma_n^{(j)}(u, v)$ are given by

$$\sigma_n^{(1)}(u, v) = \frac{1}{2} \sum_{i \in A \cap B} \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right)\{p_{n,i}(u) + p_{n,i}(v)\}, \tag{3.10}$$

$$\sigma_n^{(2)}(u, v) = \frac{1}{2} \sum_{i \notin A,\, i \notin B} \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right)\{p_{n,i}(u) + p_{n,i}(v)\}, \tag{3.11}$$

$$\sigma_n^{(3)}(u, v) = \frac{1}{2} \sum_{i \notin A,\, i \in B} \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right) p_{n,i}(u), \tag{3.12}$$

$$\sigma_n^{(4)}(u, v) = \frac{1}{2} \sum_{i \in A,\, i \notin B} \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right) p_{n,i}(v), \tag{3.13}$$

$$\sigma_n^{(5)}(u, v) = \frac{1}{2} \sum_{i \notin A,\, i \in B} \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right) p_{n,i}(v), \tag{3.14}$$

$$\sigma_n^{(6)}(u, v) = \frac{1}{2} \sum_{i \in A,\, i \notin B} \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right) p_{n,i}(u). \tag{3.15}$$

If we denote by φ_1 the restriction of the function φ to the set $\left\{\left(\frac{i}{n}, \frac{i}{n}\right) : i \in A \cap B\right\}$ we deduce that, according to the equality (3.2), for an arbitrarily chosen $\varepsilon > 0$ and n sufficiently large, we have $\left|\varphi_1\left(\frac{i}{n}, \frac{i}{n}\right)\right| < \varepsilon$. From this relation and from the obvious inequality $|x - y| \leqslant 1$ for $x, y \in [0,1]$ we can estimate $\sigma_n^{(1)}(u, v)$ in the following way

$$|\sigma_n^{(1)}(u, v)| \leqslant \frac{\varepsilon}{2} \sum_{i \in A \cap B} \left| u - \frac{i}{n} \right| \left| v - \frac{i}{n} \right| \{p_{n,i}(u) + p_{n,i}(v)\} \leqslant$$

$$\leqslant \frac{\varepsilon}{2} \left\{ \sum_{i \in A} \left| u - \frac{i}{n} \right| p_{n,i}(u) + \sum_{i \in B} \left| v - \frac{i}{n} \right| p_{n,i}(v) \right\}.$$

But the following successive inequalities are true :

$$\sum_{i \in A} \left| u - \frac{i}{n} \right| p_{n,i}(u) \leqslant \sum_{i=0}^{n} \left| u - \frac{i}{n} \right| p_{n,i}(u) \leqslant \frac{1}{2\sqrt{n}}.$$

The last inequality was proved by T. Popoviciu in 1942 (it also follows from a striking result due to Schurer and Steutel [9]). Hence, for every sufficiently large n, we have

$$|\sigma_n^{(1)}(u, v)| \leqslant \frac{\varepsilon}{2\sqrt{n}} \tag{3.16}$$

Now we evaluate the sum $\sigma_n^{(2)}$ given by (3.11).

From the equality (3.1), we get

$$\left| \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right) \right| \leqslant$$

$$\leqslant \left| f\left(\frac{i}{n}, \frac{i}{n}\right) - f\left(\frac{i}{n}, v\right) - f\left(u, \frac{i}{n}\right) + f(u, v) \right| + |f'(u, v)|$$

Because the function f is B-continuous at $[0,1]^2$, there is an $M_1 > 0$ such that

$$\left| f\left(\frac{i}{n}, \frac{i}{n}\right) - f\left(\frac{i}{n}, v\right) - f\left(u, \frac{i}{n}\right) + f(u, v) \right| \leqslant M_1$$

(see Badea [1, Lemma)]. Hence there is an $M = M_1 + |f'(u, v)|$ such that

$$\left| \varphi\left(\frac{i}{n}, \frac{i}{n}\right)\left(\frac{i}{n} - u\right)\left(\frac{i}{n} - v\right) \right| \leqslant M. \tag{3.17}$$

Using the above inequality (3.17) we can write

$$|\sigma_n^{(2)}(u, v)| \leqslant \frac{M}{2} \sum_{i \notin A,\, i \notin B} \{p_{n,i}(u) + p_{n,i}(v)\} \leqslant \frac{M}{2} \left\{ \sum_{i \notin A} p_{n,i}(u) + \sum_{i \notin B} p_{n,i}(v) \right\} \tag{3.18}$$

It is known [8] that if $\delta > 0$ and $\Delta(x) = \left\{ k : \left| x - \frac{k}{n} \right| \geqslant \delta \right\} \subset \{0, 1, \ldots\}$ then for every $m = 0, 1, 2, \ldots$ there is a universal constant $K_{2m} > 0$, such that

$$\sum_{k \in \Delta(x)} p_{n,k}(x) \leqslant \frac{K_{2m}}{n^m \delta^{2m}}$$

Using this result with $\delta = n^{-1/4}$ and $m = 3$ we find that

$$\sum_{i \notin A} p_{n,i}(u) \leqslant \frac{K_6}{n\sqrt{n}} \tag{3.19}$$

From (3.18) and (3.19), it follows that

(3.20) $$\sqrt{n}\,|\sigma_n^{(2)}(u, v)| \leq \frac{M_2}{n},$$

where $M_2 > 0$ is a universal constant.

In a similar way, we deduce that

(3.21) $$\sqrt{n}\,|\sigma_n^{(j)}(u, v)| \leq \frac{M_j}{n},\ j \in \{3, 4\},$$

where M_3, $M_4 > 0$ are also universal constants.

From inequalities (3.16), (3.20) and (3.21), we conclude that for every sufficiently large n we have

$$\sqrt{n}\left|\sum_{i=1}^{4} \sigma_n^{(i)}(u, v)\right| \leq \frac{\varepsilon}{2} + \frac{M_5}{n}$$

($M_5 > 0$ is a universal constant), i.e. we get

(3.22) $$\sum_{i=1}^{4} \sigma_n^{(i)}(u, v) = o\left(\frac{1}{\sqrt{n}}\right)$$

On the other hand, using inequality (3.17), we can write

(3.23) $$|\sigma_n^{(5)}(u, v)| \leq \frac{M}{2} \sum_{i \notin A,\ i \in B} p_{n,i}(v) \leq \frac{M}{2} \sum_{i \notin A} p_{n,i}(v) \leq$$

$$\leq \frac{M\sqrt{n}}{2} \sum_{i=0}^{n} \left(u - \frac{i}{n}\right)^2 p_{n,i}(v)$$

Using again the identities known from the theory of the Bernstein polynomials we get

(3.24) $$\sum_{i=0}^{n} \left(u - \frac{i}{n}\right)^2 p_{n,i}(v) = (u - v)^2 + \frac{v(1 - v)}{n}$$

From (3.23) and (3.24) if follows that

(3.25) $$\sigma_n^{(5)}(u, v) = O\left(\frac{1}{\sqrt{n}}\right) + O(\sqrt{n}).$$

In a similar way, we conclude that

(3.26) $$\sigma_n^{(6)}(u, v) = O\left(\frac{1}{\sqrt{n}}\right) + O(\sqrt{n}).$$

From relations (3.9), (3.22), (3.25) and (3.26), we get the following estimation

(3.27) $$\sigma_n(u, v) = o\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{1}{\sqrt{n}}\right) + O(\sqrt{n}).$$

From relations (3.3), (3.8) and (3.27) we have the desired result. Now the proof of Theorem 3 is complete.

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