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A K-MONOTONE BEST APPROXIMATION OPERATOR WHICH IS NEITHER MONOTONE AND (ESSENTIALLY) NOR (O)-MONOTONE

RADU PRECUP
(Cluj-Napoca)

1. Let X and Y be two linear spaces, $\langle .,. \rangle : X \times Y \to \mathbb{R}$ a bilinear functional, $K \subset X$ a convex cone (i.e. $K + K \subset K$ and $\alpha K \subset K$ for any $\alpha \geqslant 0$), K^* the polar cone of K, that is $K^* = \{y \in Y;\ \langle x, y \rangle \geqslant 0$ for all $x \in K\}$ and let A be an operator from X into Y.

The operator A is called *monotone* if

(1) $$\langle x - x', Ax - Ax' \rangle \geqslant 0 \text{ for all } x, x' \in X$$

and is said to be (O) – *monotone* if

(2) $$Ax - Ax' \in K^* \text{ for all } x, x' \in X \text{ such that } x - x' \in K.$$

In our previous paper [3] we have defined and studied the larger class of the K-monotone operators.

Definition ([3]). *The operator* $A : X \to Y$ *is said to be* K-*monotone if*

(3) $\langle x - x', Ax - Ax' \rangle \geqslant 0$ *for all* $x, x' \in X$ *such that* $x - x' \in K$.

The monotonicity, (o)-monotonicity and K-monotonicity properties can be extended to multivalued mappings.

Obviously, any monotone or (o)-monotone mapping is a K-monotone one. Therefore, the study of the K-monotone mappings is important for a synthesis of some results in monotone and in (o)-monotone mappings.

In the papers [3], [4] we have proved that if the convex cone K of a reflexive Banach space X has non-empty interior, then some well-known results in the monotone operators theory, about maximality, local boundedness and demicontinuity can be extended for K-monotone mappings from X into its dual space X^*. Moreover, if K has non-empty interior with respect to the weak topology on the reflexive Banach space X, then each K-monotone, K-hemicontinuous and coercive operator A from X into X^* is surjective (A is said to be K-hemicontinuous provided that for every $x \in X$ and $u \in K$ the map $t \mapsto A(x + tu)$, $t \in \mathbb{R}$, is continuous in origin with respect to $\sigma(X^*, X)$-topology of X^*).

The aim of this paper is to give an example of K-monotone operator which is neither monotone and (essentially) nor (o)-monotone.

2. In what follows X is a real Hilbert space, $Y = X$ and the bilinear functional $\langle .,. \rangle$ is the scalar product on X. Denote by $\|\cdot\|$ the norm in X induced by the scalar product, by $\|\cdot\|'$ an arbitrary norm in X, equivalent with $\|\cdot\|$ and by C a nonvoid closed convex subset of X.

The mapping $B = B(\|\cdot\|'; C)$, $B : X \to 2^X$ assigning to each $x \in X$ the subset $\{y \in C;\ \|x - y\|' = \inf_{z \in C} \|x - z\|'\}$ is called the best approximation mapping by elements of C and with respect to the norm $\|\cdot\|'$.

Since the space $(X, \|\cdot\|')$ is reflexive, we have $Bx \neq \emptyset$ for all $x \in \in X$. Moreover, if the norm $\|\cdot\|'$ or the subset C is strictly convex, then for each $x \in X$ the subset Bx contains a unique element denoted by x^*; in this case the mapping B is an operator from X into X.

It is well known that the operator $B(\|\cdot\|; C)$ is monotone for every nonvoid closed convex subset C.

For other norms $\|\cdot\|'$ the operators $B(\|\cdot\|'; C)$ can be not monotone, but in some additional geometrical conditions imposed to the norms and to the subsets C, they can be (o)-monotone or only K-monotone with respect to certain cones K.

In what follows let us consider the strictly convex set $C = \{x \in X;\ \|x\| \leq 1\}$, a fixed element $u \in X$, $\|u\| = 1$ and the norm $\|\cdot\|_u$ associated to u as follows:

(4) $$\|x\|_u = \|x\|, \text{ if } |\langle x, u\rangle| \geq \|x\|/\sqrt{2}$$
$$= \|x_u\|\sqrt{2}, \text{ if } |\langle u, x\rangle| < \|x\|/\sqrt{2},$$

where $x_u = x - \langle x, u\rangle u$.

If we put $v = x_u/\|x_u\|$, then we remark that:

(5) $$\|x\|_u = \|x\|, \text{ if } \langle x, v\rangle \leq \|x\|/\sqrt{2}$$
$$= \langle x, v\rangle\sqrt{2}, \text{ if } \langle x, v\rangle > \|x\|/\sqrt{2}.$$

Let us consider the following convex cone:

(6) $$K_u = \{x \in X;\ \langle x, u\rangle \geq \|x\|/\sqrt{2}\}.$$

If we assume that $\dim X \geq 2$ and we write A in short instead of $B(\|\cdot\|_u; C)$, then we can get the following main result.

THEOREM. 1°. *The operators* A and $-A$ *are not monotone.*

2°. *For any convex cone* $K \subset X$ *with* $\dim K \geq 2$, *the operators* A *and* $-A$ *are not* (o)*-monotone.*

3°. *The operator* A *is* K_u *– monotone.*

3. For proof we need some lemmas:

LEMMA 1. *For any* $x \in X$, *the best approximation element* x^* *of* x *by elements of* C, *with respect to the norm* $\|\cdot\|_u$, *belongs to span* $\{x, u\}$.

Proof. Suppose that for some $x \in X$, the best approximation element x^* of x by elements of C and with respect to the norm $\|\cdot\|_u$, does not belong to span $\{x, u\} = X_x$. Then we may find $x^{**} \in$ span $\{x, u, x^*\}$ such that $x^{**} \neq x^*$, $\langle u, x^* - x^{**}\rangle = \langle x, x^* - x^{**}\rangle = 0$ and $d(x^*,$

$X_x) = d(x^{**}, X_x)$, where for some $z \in X$ we denote $d(z, X_x) = \inf\{\|z - -y\|;\ y \in X_x\}$. It follows that $\|x^{**}\| = \|x^*\| = 1$, $\|x - x^{**}\| = \|x - x^*\|$ and $\langle u, x - x^{**}\rangle = \langle u, x - x^*\rangle$. Therefore, $\|x - x^{**}\|_u = \|x - x^*\|_u$, which contradicts the uniqueness of the best approximation element by elements of the strictly convex set C, with respect to the norm $\|\cdot\|_u$.

LEMMA 2. *Let* $x \in X$, $\|x\| > 1$.

1°. *If* $x \in K_u \cup (-K_u)$, *then* $x^* = x/\|x\|$.

2°. *If* $x \notin K_u \cup (-K_u)$ *and* $x - x_u/\|x_u\| \notin K_u \cup (-K_u)$, *then* $x^* = = x_u/\|x_u\|$.

3°. *If* $x \notin K_u \cup (-K_u)$ *and* $x - x_u/\|x_u\| \in K_u \cup (-K_u)$, *then* x^* *is the unique element satisfying the following conditions:*

$$x^* = (\alpha\lambda/\|x_u\|)\, x_u + (1 - \alpha)x \text{ for certain } \alpha,\ \lambda \in [0, 1],$$

(7) $$|\langle u, x - x^*\rangle| = \|x - x^*\|/\sqrt{2},$$

$$\|x^*\| = 1.$$

Proof. 1°. Let $x \in K_u \cup (-K_u)$, $\|x\| > 1$. Then $x - x/\|x\| \in K_u \cup (-K_u)$ and therefore $\|x - x/\|x\|\|_u = \|x - x/\|x\|\|$. Since $\|x - x/\|x\|\| \leq \|x - y\|$ for every $y \in C$ with $\|y\| = 1$, as we can easily see, and since $\|x - y\| \leq \|x - y\|_u$, one obtains $\|x - x/\|x\|\|_u \leq \|x - y\|_u$ for every $y \in C$ with $\|y\| = 1$.

2.° In what follows we denote by v the element $x_u/\|x_u\|$. From $x - - v \notin X_u \cup (-K_u)$ we deduce that $x - v = \lambda z$, where $\lambda > 0$, $\|z\| \leq 1$ and $\langle v, z\rangle = 1/\sqrt{2}$. We have $\|z\|_u = 1$ and $\|x - v\|_u = \lambda$. Now we fix any $\bar{v} \in$ span $\{x, u\} \cap C$. If we denote $\bar{\lambda} = \|x - \bar{v}\|_u$, then we have

$$\bar{\lambda}/\sqrt{2} \geq |\langle v, x - \bar{v}\rangle| = |\langle v, v - \bar{v} + \lambda z\rangle| =$$

$$= |1 + \lambda/\sqrt{2} - \langle v, \bar{v}\rangle| \geq |1 + \lambda/\sqrt{2} - |\langle v, \bar{v}\rangle|\ | \geq \lambda/\sqrt{2}$$

because $|\langle v, \bar{v}\rangle| \leq 1$. Thus $\bar{\lambda} \geq \lambda$ and since $\bar{v}$ was arbitrary in span $\{x, u\} \cap \cap C$, it follows by Lemma 1 that $x^* = v$.

3°. Applying Lemma 1 we will look for x^* in the two-dimensional subspace $X_x =$ span $\{x, u\}$ equipped with the orthonormal basis (v, u). Using coordinates we may write $x = (\|x\|\cos\theta, \|x\|\sin\theta)$, $\|x\| > 1$, $-\pi < < \theta \leq \pi$.
Since $v = x_u/\|x_u\|$ we can see that $|\theta| \leq \pi/2$ and taking into account the inequality $|\langle u, x\rangle| < \|x\|/\sqrt{2}$ we get that $|\theta| < \pi/4$. Also, $\theta \neq 0$. Indeed, otherwise (i.e. $\theta = 0$) we should have $x = \|x\|v$ and consequently $x - v = (\|x\| - 1)v \notin X_u \cup (-K_u)$, a contradiction. Therefore, $\theta \neq 0$. Replacing, if necessary, the basis (v, u) by $(v, -u)$ we may assume, without loss of generality, that $0 < \theta < \pi/4$. On the other hand, from $x - v \in K_u \cup (-K_u)$ we easily find that

(8) $$\cos\theta - \sin\theta \leq 1/\|x\|.$$

Since each element $z \in X_x$, $\|z\| = 1$ can be uniquely written in the form $z = x - \lambda y$, where $\lambda > 0$, $y \in X_x$ and $\|y\| = 1$, we will look for x^* in the form $x^* = x - \lambda(y^*)y^*$, where $\lambda(y^*) > 0$, $y^* \in X_x$, $\|y^*\| = 1$. Then

$\|x - x^*\|_u = \lambda(y^*)\|y^*\|_u$ and so it remains to solve the following problem of minimization in X_x of the function

(9) $$\lambda(y)\|y\|_u, \; y \in X_x$$

with the constraints

(10) $$\|x - \lambda(y)y\| = 1, \quad \lambda(y) > 0, \quad \|y\| = 1.$$

Let us remark that constraints (10) are equivalent to

(11) $$\lambda^2(y) - 2\langle x, y\rangle\lambda(y) + \|x\|^2 - 1 = 0, \quad \lambda(y) > 0, \; \|y\| = 1.$$

Since $\|x\|^2 - 1 > 0$ it follows that the necessary and sufficient conditions in order that the equation in $\lambda(y)$ from (11) has a positive solution, are:

(12) $$\langle x, y\rangle^2 - \|x\|^2 + 1 \geqslant 0, \quad \langle x, y\rangle > 0.$$

In these conditions the minimum value of $\lambda(y)$ is

(13) $$\lambda(y) = \langle x, y\rangle - \sqrt{\langle x, y\rangle^2 - \|x\|^2 + 1} = \|x\| \cos(t - \theta) - \sqrt{1 - \|x\|^2 \sin^2(t - \theta)} = (\|x\|^2 - 1)/(\|x\|\cos(t - \theta) + \sqrt{1 - \|x\|^2\sin^2(t - \theta)}),$$

where $(\cos t, \sin t)$, $-\pi < t \leqslant \pi$, are the coordinates of y.

We will prove that the unique solution of the problem (9)–(10) is obtained for $y^* = (1/\sqrt{2}, 1/\sqrt{2})$, that is

(14) $\lambda(y)\|y\|_u > \lambda(y^*)\|y^*\|_u$ for every $t \in (-\pi, \pi]$, $t \neq \pi/4$.

Let us denote for simplicity $a = \|x\|$. Then the system of inequalities (12) is equivalent to

(15) $$\cos(t - \theta) \geqslant \sqrt{a^2 - 1}/a$$

which gives

(16) $$\theta - \arccos \sqrt{a^2 - 1}/a \leqslant t \leqslant \theta + \arccos \sqrt{a^2 - 1}/a.$$

To prove (14) we will consider more cases:

Case 1. Let $\theta - \arccos \sqrt{a^2 - 1}/a \leqslant t \leqslant -\pi/4$. Since $0 < \theta < \pi/4$ we have $|\cos t| \leqslant 1/\sqrt{2}$, that is $|\langle y, v\rangle| \leqslant 1/\sqrt{2}$. So, according to (5), $\|y\|_u = \|y\| = 1 = \|y^*\|_u$. On the other hand $0 < \cos(\theta - t) < \cos(\pi/4 - \theta)$ and using (13) we see that $\lambda(y) > \lambda(y^*)$. Therefore, in this case, relation (14) holds.

Case 2. Let $-\pi/4 < t < \pi/4$. Then $\cos t = \langle y, v\rangle > 1/\sqrt{2}$ and consequently, by (5), $\|y\|_u = \sqrt{2} \cos t$, while $\|y^*\|_u = 1 = \sqrt{2}\cos \pi/4$. Now, let us denote

$f(t) = (a \cos(t - \theta) - \sqrt{a^2\cos^2(t - \theta) - a^2 + 1}) \cos t$. According to (13) we see that for $-\pi/4 < t < \pi/4$ inequality (14) is equivalent to

(17) $$f(t) > f(\pi/4).$$

To prove (17) we will consider two subcases:

Subcase 2a. Let $-\pi/4 < t < \theta$. We will prove the inequality

(18) $$f(t) > f(\theta),$$

or equivalently the following one

(19) $$a \sin t \sin(\theta - t) + \cos \theta > \cos t \sqrt{a^2\cos^2(t - \theta) - a^2 + 1}$$

for $-\pi/4 < t < \theta$.

First we note that the left-hand side of expression (19) is nonnegative. This is clear for $0 \leqslant t < \theta$. Now, if we suppose that $-\pi/4 < t < 0$ and we take into account that $0 < \sin(\theta - t) \leqslant 1/a$ (in view of (15)), then we obtain $\sin t \leqslant a \sin t \sin(\theta - t) < 0$. But, $\cos \theta + \sin t > 0$ because $0 < \theta < \pi/4$ and $-\pi/4 < t < 0$. It follows that the left-hand side of (19) is nonnegative even for $-\pi/4 < t < \theta$.

Consequently, we may use the standard technique of eliminating the radicals and so, inequality (19) can be reduced to

$$a \sin^2(\theta - t) + \sin(\theta - t) \sin(\theta + t) > 0$$

which is fulfilled for $-\pi/4 < t < \theta$, as we can easily see.

Subcase 2b. Let $\theta \leqslant t < \pi/4$. We will prove that $f'(t) < 0$ for every $t \in [\theta, \pi/4)$, that is

$$a^2\sin(t - \theta) \cos(2t - \theta) + \sin t < a\sin(2t - \theta) \sqrt{1 - a^2\sin^2(t - \theta)}$$

After some simple transformations this inequality can be reduced to

$$\cos \theta - \text{ctg}\, t \sin \theta < 1/a$$

which according to (8) is satisfied because $\text{ctg}\, t > \text{ctg}\, \pi/4 = 1$. Thus, $f'(t) < 0$ for $\theta \leqslant t < \pi/4$ and in consequence

(20) $$f(t) > f(\pi/4) \text{ for every } t \in [\theta, \pi/4).$$

Now from relations (18) and (20) we may infer that inequality (17) is true for every $t \in (-\pi/4, \pi/4)$.

Case 3. Let $\pi/4 < t \leqslant \theta + \arccos \sqrt{a^2 - 1}/a$. From $0 < \pi/4 - \theta < t - \theta \leqslant \arccos \sqrt{a^2 - 1}/a < \pi/2$, it follows that $\cos(t - \theta) < \cos(\pi/4 - \theta)$ which, by (13), yields $\lambda(y) > \lambda(y^*)$. On the other hand, since $\pi/4 < t < \pi/4 + \pi/2$ we have $|\langle y, v\rangle| = |\cos t| < 1/\sqrt{2}$. Thus, by (5), one has $\|y\|_u = \|y\| = 1 = \|y^*\|_u$. Therefore, relation (14) holds.

Finally, it is easy to see that $x^* = x - \lambda(y^*)y^*$ is the unique solution of system (7). The proof of Lemma 2 is now complete.

Proof of Theorem. 1°. Let $x \in X$ such that $\|x\| = 1$ and $\langle x, u\rangle = -1/\sqrt{2}$. We set $x_\lambda = v + \lambda x$, where $\lambda > 0$. It is easy to see that $x_\lambda \notin K_u \cup (-K_u)$ and $x_\lambda - v \in K_u \cup (-K_u)$. According to Lemma 2.3° we have $Ax_\lambda = v$. On the other hand $Ax = x$. Therefore,

$$\langle x_\lambda - x, \; Ax_\lambda - Ax\rangle = \langle v + (\lambda - 1)\, x, \; v - x\rangle = (1 - 1/\sqrt{2})(2 - \lambda).$$

This number is positive for $0 < \lambda < 2$ and negative for $\lambda > 2$. Hence, the operators A and $-A$ are not monotone.

2°. Let K be a convex cone of X with $\dim K \geqslant 2$. We will consider the following cases:

Case 1. Assume that $\operatorname{ri} K \cap (\operatorname{int} K_u \cup (-\operatorname{int} K_u)) \neq \emptyset$, where by $\operatorname{int} K_u$ and $\operatorname{ri} K$ we have denoted the interior of K_u, respectively the relative interior of K.
Then there are $x, y \in K$ with $x \neq y$, $\|x\| = \|y\| = 1$, such that $x, y \in K_u$ or $x, y \in (-K_u)$. It follows that $x + y \in K_u \cup (-K_u)$ and $\|x + y\| > 1$, hence, applying Lemma 2.1°, $A(x + y) = (x + y)/\|x + y\|$.
Also, $Ax = x$. Now, it is easy to see that
$\langle x, (x + y)/\|x + y\| - x\rangle < 0$ and $\langle y, -(x + y)/\|x + y\|\rangle < 0$, that is $A(x + y) - Ax \notin K^*$ and $-A(x + y) - (-Ax) \notin K^*$, although $(x + y) - x = y \in K$. Hence, the operators A and $-A$ are not (o) – monotone with respect to K and K^*.

Case 2. Assume that there exists $x \in K$, with $\|x\| = 1$, such that $0 < |\langle x, u\rangle| < 1/\sqrt{2}$. Then, we may find $\lambda > 1$ in order that the element $x_\lambda = \lambda x$ satisfies $x_\lambda - v \notin K_u \cup (-K_u)$. It follows that $Ax_\lambda = v$. Since

$$x_\lambda - x = (\lambda - 1)x \in K \text{ and } Ax_\lambda - Ax = v - x \notin K^*$$

because $\langle x, v - x\rangle = \langle x, v\rangle - 1 < 0$, we conclude that A is not (o)-monotone.

If in addition $\langle x, u\rangle > 0$, then we may find $\mu > 0$ such that the element $y = u + \mu x$ satisfies $\langle y, u\rangle = \|y\|/\sqrt{2}$. We have $Ay = y/\|y\|$ and $Au = u$. Since $y - u = \mu x \in K$ and $-Ay - (-Au) = -y/\|y\| + u \notin K^*$ because as we can easily see $\langle x, -y/\|y\| + u\rangle < 0$, we may conclude that $-A$ is not (o)-monotone.

If $\langle x, u\rangle < 0$, then one proceeds in the same way taking $y = u - \mu x$, with $\mu > 0$.

Case 3. Suppose that $\langle z, u\rangle = 0$ for every $z \in K$. Since $\dim K \geqslant 2$, we may find $x, y \in K$, such that $x \neq y$, $\|x\| = \|y\| = 1$ and $\|x + y\| > 1$. Applying Lemma 2.2° we obtain $A(x + y) = (x + y)/\|x + y\|$. Also, $Ax = x$. Now, we easily observe that $\langle x, (x + y)/\|x + y\| - x\rangle < 0$ and $\langle y, -(x + y)/\|x + y\| + x\rangle < 0$ and since $(x + y) - x = y \in K$ we may infer that A and $-A$ are not (o)-monotone.

Thus the second part of Theorem is proved.

3°. Let $x, y \in X$, such that $x \neq y$, $x - y \in K_u \cup (-K_u)$ and $Ax \neq Ay$. We will prove that

$$\langle x - y, Ax - Ay\rangle \geqslant 0. \tag{21}$$

Assume for the beginning that $\|x\| \geqslant 1$ and $\|y\| \geqslant 1$. We will consider more cases:

Case 1. Let $x, y \in K_u \cup (-K_u)$. Then $Ax = x/\|x\|$, $Ay = y/\|y\|$ and we immediately see that relation (21) is fulfilled.

Case 2. Suppose that $x, y \notin K_u \cup (-K_u)$ and $x - x_u/\|x_u\|$, $y - y_u/\|y_u\| \notin K_u \cup (-K_u)$. Then, by Lemma 2, $Ax = x^* = x_u/\|x_u\|$ and $Ay = y^* = y_u/\|y_u\|$.

In the 3-dimensional subspace span $\{x^*, y^*, u\}$ we may use the following coordinatizations: $x^* = (1, 0, 0)$; $u = (0, 0, 1)$; $y^* = (\cos t, \sin t, 0)$, $t \in [0, 2\pi)$; $x = (a, 0, \alpha(a - 1))$, $a \geqslant 1$, $|\alpha| \leqslant 1$; $y = (b \cos t, b \sin t, \beta(b - 1))$, $b \geqslant 1$, $|\beta| \leqslant 1$. Then, we obtain $\langle x - y, Ax - Ay\rangle = (a + b)(1 - \cos t) \geqslant 0$, which proves (21).

Case 3. Let $x, y \notin K_u \cup (-K_u)$ such that $x - x_u/\|x_u\| \in K_u$ and $y - y_u/\|y_u\| \notin K_u \cup (-K_u)$. Then, $Ay = y^* = y_u/\|y_u\|$ and $Ax = x^*$, where x^* is the solution of system (7). In the subspace span$\{x^*, y^*, u\}$ we may use the following coordinates: $x^* = (\cos t, 0, \sin t)$, $0 \leqslant t \leqslant \pi/4$;

$$y^* = (\cos \varphi, \sin \varphi, 0), \quad 0 \leqslant \varphi \leqslant \pi;$$

$x = (\cos t + a, 0, \sin t + a)$, $a \geqslant 0$; $y = ((1 + b)\cos \varphi, (1 + b) \sin \varphi, \alpha b)$, $b \geqslant 0$, $|\alpha| \leqslant 1$. Also we denote $y_1^* = (\cos \varphi, 0, 0)$.
We have $\langle x - y, x^* - y^*\rangle = 2(1 - \cos t \cos \varphi) + a(\cos t + \sin t - \cos \varphi) + b(1 - \cos t \cos \varphi - \alpha \sin t)$.
Since $\cos t + \sin t - \cos \varphi \geqslant \cos t + \sin t - 1 \geqslant 0$ we observe that if $1 - \cos t \cos \varphi - \alpha \sin t \geqslant 0$, then (21) is satisfied.

Next we will suppose that $1 - \cos t \cos \varphi - \alpha \sin t < 0$. Then, $\alpha > 0$ and

$$0 \leqslant 1 - \cos t \cos \varphi < \sin t. \tag{22}$$

We have $\langle x - y, x^* - y^*\rangle = \langle x - y, x^* - y_1^*\rangle + (1 + b)\sin^2\varphi$.
Therefore, $\langle x - y, x^* - y^*\rangle \geqslant \langle x - y, x^* - y_1^*\rangle$ and so, to prove (21) it is sufficient to show that $\langle x - y, x^* - y_1^*\rangle \geqslant 0$. For this, we will prove first that $x^* - y_1^* \in K_u = K_u^*$ and after, that $x - y \in K_u$.

Indeed, using (22), we deduce that $\sin^2 t > 1 - 2 \cos t \cos \varphi + \cos^2 t \cos^2 \varphi \geqslant \cos^2 t + \cos^2 \varphi - 2 \cos t \cos \varphi$.
Therefore, $\sin t \geqslant ((\cos t - \cos \varphi)^2 + \sin^2 t)^{1/2}/\sqrt{2}$, that is $x^* - y_1^* \in K_u$, where we have taken into account that $x^* - y_1^* = (\cos t - \cos \varphi, 0, \sin t)$.

Next, we have to show that $x - y \notin (-K_u)$. Suppose that this is not the case, i.e. $x - y \in (-K_u)$. Then,

$\alpha b - \sin t - a \geqslant \|x - y\|/\sqrt{2}$, which is equivalent to the system consisting of the following inequality

$$b \geqslant (\sin t + a)/\alpha \tag{23}$$

and of the inequality obtained after the elimination of radicals

$$f(b) = b^2(1 - \alpha^2) + 2b(1 - \cos t \cos \varphi - a \cos \varphi + \alpha \sin t + a\alpha) + 2 \cos^2 t + 2a(\cos t - \sin t) - 2 \cos \varphi (\cos t + a) \leqslant 0. \tag{24}$$

To derive a contradiction we will prove that this system in b has no solution.

If $\alpha = 1$, then since $1 - \cos t \cos \varphi - a \cos \varphi + \sin t + a > 0$ and $f(\sin t + a) > 0$, it is clear that system (23)–(24) has no solution.

Next, let $0 < \alpha < 1$. We have $f((\sin t + a)/\alpha) = a^2(1 - 2\alpha \cos\varphi + \alpha^2) + 2a(\alpha \cos t(1 - \cos \varphi) + (\alpha + \sin t)(1 - \alpha \cos \varphi) - \alpha(1 - \alpha) \cos t) + \sin^2 t + 2\alpha \sin t(1 - \cos t \cos \varphi) + \alpha^2(1 - 2 \cos t \cos \varphi + \cos^2 t)$.

Since $(\alpha + \sin t)(1 - \alpha \cos \varphi) > \alpha(1-\alpha) \geqslant \alpha(1-\alpha)\cos t$, we infer that $f((\sin t + a)/\alpha) > 0$.
On the other hand, the demi-sum of the roots of $f(b)$ does not exceed $(\sin t + a)/\alpha$, as we can easily see.
Therefore, $f(b) > 0$ for every b satisfying (23).
Consequently, system (23)—(24) has no solution. This contradiction shows that $x - y \notin (-K_u)$.

Case 4. Let us assume that $x \in K_u$, $y \notin K_u \cup (-K_u)$ and $y - y_u/\|y_u\| \notin$ $\notin K_u \cup (-K_u)$.
Then, we have $Ax = x^* = x/\|x\|$ and $Ay = y^* = y_u/\|y_u\|$. Passing to coordinates in a 3-dimensional subspace of X containing x^*, y^* and u, we may write:

$$u = (0, 0, 1);\ x^* = (\cos t, 0, \sin t),\ \pi/4 \leqslant t \leqslant \pi/2;$$

$$y^* = (\cos\varphi, \sin\varphi, 0),\ 0 \leqslant \varphi \leqslant \pi;\ x = (a\cos t, 0, a \sin t),\ a \geqslant 1;$$

$$y = ((1+b)\cos\varphi, (1+b)\sin\varphi, \alpha b),\ b \geqslant 0,\ |\alpha| \leqslant 1.$$

Also, consider $y_1^* = (\cos\varphi, 0, 0)$. We have

$$\langle x - y, x^* - y^*\rangle = a(1 - \cos t\cos\varphi) + \\ + b(1 - \cos t \cos\varphi - \alpha \sin t) + 1 - \cos t\cos\varphi.$$

If the coefficient of b in this expression is nonnegative, then it is clear that relation (21) is satisfied. Next, let $1 - \cos t\cos\varphi - \alpha\sin t < 0$. It follows that $0 < \alpha \leqslant 1$ and (22) holds. Since $\langle x - y, x^* - y^*\rangle \geqslant$ $\geqslant \langle x - y, x^* - y_1^*\rangle$, in order that (21) be true it is sufficient to prove that $\langle x - y, x^* - y_1^*\rangle \geqslant 0$. As at Case 3, one shows that $x^* - y_1^* \in K_u = K_u^*$.
Next, the relation $x - y \in (-K_u)$ is equivalent to the system:

$$b \geqslant a \sin t/\alpha$$

$$b^2(1-\alpha^2) + 2b(1 - a\cos t\cos\varphi + a\alpha\sin t) + 1 + 2a^2\cos^2 t - \\ - a^2 - 2a\cos t\cos\varphi \leqslant 0,$$

which, as at Case 3, has no solution. Therefore, $x - y \in K_u$.

Case 5. Suppose that $x \in K_u$, $y \notin K_u \cup (-K_u)$ and $y - y_u/\|y_u\| \in$ $\in K_u \cup (-K_u)$. Then $Ax = x/\|x\|$ and $Ay = y^*$, where y^* is the solution of (7). Now, we may use coordinates as follows: $x^* = (\cos t, 0, \sin t)$, $\pi/4 \leqslant t \leqslant \pi/2$; $u = (0, 0, 1)$; $x = (a\cos t, 0, a\sin t)$, $a \geqslant 1$;

$$y^* = (\cos\psi\cos\varphi, \cos\psi\sin\varphi, \sin\psi),\ -\pi/4 < \psi < \pi/4,\ 0 \leqslant \varphi \leqslant \pi;$$

$$y = ((\cos\psi + b)\cos\varphi, (\cos\psi + b)\sin\varphi, \sin\psi + \varepsilon b),\ b \geqslant 0,\ |\varepsilon| = 1,$$

$$\varepsilon\psi = |\psi|.\ \text{Let } y_1^* = (\cos\psi\cos\varphi, 0, \sin\psi).$$

We have

$$\langle x - y, x^* - y^*\rangle = (a+1)(1 - \cos t\cos\psi\cos\varphi - \sin t\sin\psi) + \\ + b(\cos\psi + \varepsilon\sin\psi - \varepsilon\sin t - \cos t\cos\varphi). \tag{25}$$

Since $(\cos t\cos\psi\cos\varphi + \sin t\sin\psi)^2 \leqslant (\cos^2 t + \sin^2 t)\cdot(\cos^2\psi\cos^2\varphi +$ $+ \sin^2\psi) \leqslant 1$,
we may infer that $1 - \cos t\cos\psi\cos\varphi - \sin t\sin\psi \geqslant 0$. Now, if the coefficient of b in formula (25) is nonnegative, then it is obvious that (21) is true. Next, we assume that

$$\cos\psi + \varepsilon\sin\psi - \varepsilon\sin t - \cos t\cos\varphi < 0. \tag{26}$$

Let us remark that relation (26) is possible only for $\varepsilon = +1$.
It suffices, in this case again, to justify the inequality $\langle x - y, x^* - y_1^*\rangle \geqslant$ $\geqslant 0$ and for this, to show that
$x^* - y_1^* \in K_u$ and $x - y \notin (-K_u)$. The first of these relations is equivalent to $\sin t - \sin\psi \geqslant \|x^* - y_1^*\|/\sqrt{2}$, and, after elimination of radicals, to the system of inequalities: $\sin t \geqslant \sin\psi$, $(\sin t - \sin\psi)^2 \geqslant (\cos t -$ $- \cos\psi\cos\varphi)^2$. The first inequality is obvious; to verify the second one it suffices, by (26), that $(\cos\psi - \cos t\cos\varphi)^2 \geqslant (\cos t - \cos\psi$ $\cos\varphi)^2$ or equivalently $\cos^2\psi - \cos^2 t \geqslant \cos^2\varphi(\cos^2\psi - \cos^2 t)$, which is clear.
Therefore, $x^* - y_1^* \in K_u = K_u^*$.

Now, assume that $x - y \in (-K_u)$. Then, it follows that $b \geqslant a\sin t -$ $- \sin\psi$ and

$$(\sin\psi + b - a\sin t)^2 \geqslant (a\cos t - \cos\psi\cos\varphi - b\cos\varphi)^2 + \\ + (\cos\psi + b)^2\sin^2\varphi.$$

This last inequality can be written equivalently:

$$f(b) = 2b(\sin\psi - \cos\psi - a\sin t + a\cos t\cos\varphi) + \\ + (\sin\psi - a\sin t)^2 - (a\cos t - \cos\psi\cos\varphi)^2 - \cos^2\psi\sin^2\varphi \geqslant 0.$$

Observing that the coefficient of b is negative and that $f(a\sin t - \sin\psi) <$ < 0, one derives a contradiction.
Consequently, $x - y \notin (-K_u)$.

Case 6. Let $x, y \notin K_u \cup (-K_u)$ such that

$$x - x_u/\|x_u\| \in K_u \cup (-K_u) \text{ and } y - y_u/\|y_u\| \in K_u \cup (-K_u).$$

Now, one has $Ax = x^*$ and $Ay = y^*$, where x^* and y^* are given by (7).
Using coordinates we may identify: $u = (0, 0, \rho)$, $|\rho| = 1$; $x^* = (\cos t,$ $0, \sin t)$, $0 < t < \pi/4$; $y^* = (\cos\psi\cos\varphi, \cos\psi\sin\varphi, \sin\psi)$, $|\psi| < \pi/4$, $|\psi| \leqslant t$, $0 \leqslant \varphi \leqslant \pi$; $x = (\cos t + a, 0, \sin t + a)$, $a \geqslant 0$; $y = ((\cos\psi +$ $+ b)\cos\varphi, (\cos\psi + b)\sin\varphi, \sin\psi + \varepsilon b)$, $b \geqslant 0$, $|\varepsilon| = 1$, $\varepsilon\psi = |\psi|$.
Denote $y_1^* = (\cos\psi\cos\varphi, 0, \sin\psi)$. We have

$$\langle x - y, x^* - y^*\rangle = a(\cos t + \sin t - \cos\psi\cos\varphi - \sin\psi) + \\ + b(\cos\psi + \varepsilon\sin\psi - \cos t\cos\varphi - \varepsilon\sin t) + (\cos t - \cos\psi\cos\varphi)^2 + \\ + \cos^2\psi\sin^2\varphi + (\sin t - \sin\psi)^2.$$

Since $\cos t + \sin t - \cos\psi\cos\varphi - \sin\psi \geqslant \cos t + \sin t - \cos\psi -$ $- \sin\psi = \sqrt{2}(\sin(\pi/4 + t) - \sin(\pi/4 + \psi)) \geqslant 0$, it is clear that if the coefficient of b is nonnegative, then (21) holds. Next, we suppose that

$$\cos\psi + \varepsilon\sin\psi - \cos t\cos\varphi - \varepsilon\sin t < 0. \tag{27}$$

It follows that $\varepsilon = +1$.
We will prove that $\rho(x^* - y_1^*) \in K_u$ and $\rho(x - y) \notin (-K_u)$. From these it will derive $\langle x - y, x^* - y_1^* \rangle \geqslant 0$ and finally (21).

Relation $\rho(x^* - y_1^*) \in K_u$ is equivalent to the system $\sin t \geqslant \sin \psi$, $(\sin t - \sin \psi)^2 \geqslant (\cos t - \cos \psi \cos \varphi)^2$. The first inequality is obvious. Concerning the second one, let us remark, in view of (27), that it suffices to have $(\cos \psi - \cos t \cos \varphi)^2 \geqslant (\cos t - \cos \psi \cos \varphi)^2$, that is $\cos^2 \psi - \cos^2 t \geqslant \cos^2 \varphi \, (\cos^2 \psi - \cos^2 t)$. But this inequality is clearly true.

Next, if we should have $\rho(x - y) \in (-K_u)$, then $b \geqslant a + \sin t - \sin \psi$ and $(\sin \psi + b - \sin t - a)^2 \geqslant (\cos t + a - \cos \psi \cos \varphi - b \cos \varphi)^2 + (\cos \psi + b)^2 \sin^2 \varphi$.
The last inequality is equivalent to

$$f(b) = 2b \, (\sin \psi - \cos \psi - \sin t - a + a \cos \varphi + \cos t \cos \varphi) + (\sin \psi - \sin t - a)^2 - (\cos t + a - \cos \psi \cos \varphi)^2 - \cos^2 \psi \sin^2 \geqslant 0.$$

Since $f(a + \sin t - \sin \psi) < 0$ and the coefficient of b is negative, we obtain a contradiction. Therefore, $\rho(x - y) \in K_u$ as claimed.

To complete the proof of the third part of our Theorem it would be necessary to demonstrate relation (21), in addition in the cases corresponding to $\|x\| < 1$ and $\|y\| \geqslant 1$. But since these cases can be more easily discussed, by using a similar technique, we omit the details about them.

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Liceul de Informatică
Calea Turzii 140—142
3400 Cluj-Napoca