MATHEMATICA — REVUE D'ANALYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION
L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 15, N° 2, 1986, pp. 163—165

SETS ON WHICH CONCAVE FUNCTIONS ARE AFFINE AND KOROVKIN CLOSURES

I. RAŞA
(Cluj-Napoca)

1. Let E be a locally convex Hausdorff space over reals and $X \subset E$ a compact convex set. Let $\text{Prob}(X)$ be the set of all probability Radon measures on X. If $\mu \in \text{Prob}(X)$ let $r(\mu)$ be the barycenter of μ and $c(\mu) = \text{cl}\,(\text{conv}\,(\text{supp}\,\mu))$; then $r(\mu) \in c(\mu)$ (see [4, Proposition 1.2]). Let δ_x denote the Dirac measure at $x \in X$.

THEOREM 1. *Let $\mu \in \text{Prob}\,(X)$ and let $f \in C(X)$ be a concave function. Then :*

1) $\mu(f) \leqslant f(r(\mu))$,

2) *If* $\mu(f) = f(r(\mu))$, *then f is affine on* $c(\mu)$.

Proof. 1). For proofs and generalizations of this assertion see [4, p. 25], [3, p. 275], [2].

2) We can assume without loss of generality that $r(\mu) = 0$ and $f(0) = 0$; hence $\mu(f) = 0$.

(i) Let $g \in C(X)$ and $M \subset X$. If $fg(p_1x_1 + \ldots + p_nx_n) = p_1g(x_1) + \ldots + p_ng(x_n)$ for all $n \geqslant 1$, all $p_i \geqslant 0$ with $p_1 + \ldots + p_n = 1$ and all $x_i \in M$, then g is affine on $\text{cl}(\text{conv}(M))$.

We omit the easy proof of this fact.

(ii) If $x \in \text{supp}\,\mu$, $x \neq 0$ and V is a neighborhood of x (in E), then there exist $y \in V \cap X$, $z \in X$, $y \neq z$ and $a \in (0,1)$ such that $0 = ay + (1-a)z$ and f is affine on $\text{conv}\{y, z\}$.

Indeed, we can assume that V is closed, convex and $0 \notin V$. Let $a = \mu(V \cap X)$. Then $0 < a < 1$. For each Borel set $B \subset X$ let us denote

$$\nu(B) = (1/a)\mu(B \cap V), \quad \lambda(B) = (1/(1-a))\mu(B \cap (X \setminus V)).$$

Then ν, $\lambda \in \text{Prob}(X)$, $\mu = a\nu + (1-a)\lambda$, $\nu(X \cap V) = 1$. Let y and z be the barycenters of ν and λ. We have $y \in V \cap X$, $0 = ay + (1-a)z$ and $y \neq z$. Moreover $0 = f(0) = f(ay + (1-a)z) \geqslant af(y) + (1-a)f(z) \geqslant a\nu(f) + (1-a)\lambda(f) = \mu(f) = 0$. Therefore

$$f(ay + (1-a)z) = af(y) + (1-a)f(z) \tag{1}$$

The function f is concave on $\text{conv}\,\{y, z\}$ and (1) holds for a certain $a \in (0,1)$; it follows easily that f is affine on $\text{conv}\,\{y, z\}$.

(iii) Let $x_1, \ldots, x_n \in (\text{supp}\mu) \setminus \{0\}$, $p_1, \ldots, p_n \geqslant 0$, $\Sigma\, p_i = 1$. Then $f(\Sigma p_ix_i) = \Sigma p_if(x_i)$.

Indeed, let $\varepsilon > 0$. There exists a circled convex neighborhood V of 0 such that $|f(x) - f(y)| \leqslant \varepsilon/2$ for all $x, y \in X$ with $x - y \in V$. It follows from (ii) that there exist

$$y_i \in (x_i + V) \cap X,\ z_i \in X,\ y_i \neq z_i,\ a_i \in (0,1) \text{ such that}$$

$0 = a_i y_i + (1 - a_i) z_i$ and f is affine on $\operatorname{conv}\{y_i, z_i\}$. Let $a = \min\{a_i/(1 - a_i) : i = 1, \ldots, n\}$. Then $a > 0$ and $0 \in \operatorname{conv}\{y_i, -ay_i\} \subset \operatorname{conv}\{y_i, z_i\}$. This yields

$$f(-ay_i) = -af(y_i),\ i = 1, \ldots, n \tag{2}$$

From (2) and the concavity of f we deduce

$$\Sigma p_i f(y_i) \geqslant -(1/a) f(-a\Sigma\, p_i y_i) \geqslant f(\Sigma p_i y_i) \geqslant \Sigma p_i\, f(y_i).$$

Therefore

$$f(\Sigma\, p_i y_i) = \Sigma\, p_i f\,(y_i) \tag{3}$$

From $y_i - x_i \in V$ and $\Sigma p_i y_i - \Sigma\, p_i x_i \in V$ it follows

$$|f(y_i) - f(x_i)| \leqslant \varepsilon/2,\ i = 1, \ldots, n \tag{4}$$

$$|f(\Sigma\, p_i y_i) - f(\Sigma\, p_i x_i)| \leqslant \varepsilon/2. \tag{5}$$

Now (3), (4) and (5) imply $|f(\Sigma p_i x_i) - \Sigma\, p_i f\,(x_i)| \leqslant \varepsilon$. Since this holds for each $\varepsilon > 0$, $f(\Sigma\, p_i x_i) = \Sigma\, p_i f(x_i)$

(iv) Suppose that $0 \in (X \setminus \operatorname{supp} \mu) \cup \operatorname{cl}(\operatorname{supp} \mu \setminus \{0\})$. Then $c(\mu) = \operatorname{cl}(\operatorname{conv}(\operatorname{supp}\mu)) = \operatorname{cl}(\operatorname{conv}(\operatorname{supp} \mu \setminus \{0\}))$. By (iii) and (i), f is affine on $c(\mu)$.

(v) Finally, let $0 \in \operatorname{supp} \mu$ be an isolated point of $\operatorname{supp} \mu$. Then $\mu = \nu + a\delta_0$, where $a = \mu(\{0\}) > 0$ and ν is a positive Radon measure on X, $0 \notin \operatorname{supp} \nu$. The case $a = 1$ is trivial, hence let $a < 1$. Then $\lambda = (1/(1 - a))\nu$ is in Prob (X), $r(\lambda) = 0$ and $\lambda(f) = 0$. By (iv), f is affine on $c(\lambda)$. Obviously $c(\lambda) \subset c(\mu)$. Conversely, $r(\lambda) = 0$ implies $0 \in c(\lambda)$; therefore $\operatorname{supp} \mu = \{0\} \cup \operatorname{supp} \lambda \subset c(\lambda)$ and thus $c(\mu) \subset c(\lambda)$. It follows that $c(\mu) = c(\lambda)$ and f is affine on $c(\mu)$.

2. Let now $A(X)$ be the space of all continuous affine functions on X. Let F be a family of concave functions in $C(X)$. We denote by H the uniformly closed linear subspace of $C(X)$ spanned by $A(X)$ and F. Let $\operatorname{Ch}(H)$ denote the Choquet boundary of H.

COROLLARY 1. *$\operatorname{Ch}(H)$ coincides with the set of those $x \in X$ for which whenever $y, z \in X$, $y \neq z$, $x = (1/2)(y + z)$, there exists $f \in F$ such that $f(x) > (1/2)(f(y) + f(z))$.*

Proof. Let $x \in X$ with the above property. Let $\mu \in$ Prob (X), $\mu = \delta_x$ on H. By Theorem 1, every $f \in F$ is affine on $c(\mu)$. It follows that x is an extreme point of $c(\mu)$. By a result of H. Bauer (see [4, Proposition 1.4]) $\mu = \delta_x$. Consequently $x \in \operatorname{Ch}(H)$.

The other inclusion is immediate.

COROLLARY 2 ([5]). *Suppose that F contains only one function f. Then $\operatorname{Ch}(H) = X$ iff f is strictly concave on X.*

Let H' be the topological dual of H endowed with the weak*-topology. As usual we consider the state space S of H, which is the compact convex set $S = \{\varphi \in H' : \varphi \text{ positive}, \varphi(1) = 1\}$.

If $x \in X$, let $j(x) : H \to R$, $j(x)(h) = h(x)$. Then $j : X \to j(X) \subset S$ is a homeomorphism and $H = \{a \circ j : a \in A(S)\}$ (see [1]).

Let P be the class of the compact convex subsets $A \subset X$ such that all $f \in F$ are affine on A. Let $\operatorname{Kor}(H)$ denote the Korovkin closure of H (see [1]).

THEOREM 2. *The function $g \in C(X)$ is in $\operatorname{Kor}(H)$ iff g is affine on every $A \in P$.*

Proof. Let us denote

$$T = \{(\mu, y) : \mu \in \text{Prob}\ (j(X)),\ y \in j(X),\ r(\mu) = y\}.$$

Then (see [1]):

$$g \in \operatorname{Kor}(H) \text{ iff } \mu(g \circ j^{-1}) = g \circ j^{-1}(y) \text{ for all } (\mu, y) \in T \tag{6}$$

Suppose that g is affine on every $A \in P$. Let $(\mu, y) \in T$. Let $\nu \in$ Prob (X), $\nu(u) = \mu(u \circ j^{-1})$ for all $u \in C(X)$. It is easily seen that $r(\nu) = j^{-1}(y)$ and $\nu(f) = f(r(\nu))$ for each $f \in F$. By Theorem 1, each $f \in F$ is affine on $c(\nu)$, hence $c(\nu) \in P$. It follows that g is affine on $c(\nu)$. Then $\mu(g \circ j^{-1}) = \nu(g) = g(r(\nu)) = g \circ j^{-1}(y)$. We deduce from (6) that $g \in \operatorname{Kor}(H)$.

Let now $g \in \operatorname{Kor}(H)$. Let $A \in P$, $\nu \in \operatorname{Prob}(A)$; we have to prove that $\nu(g) = g(r(\nu))$. Let $\mu \in \operatorname{Prob}(j(X))$, $\mu(v) = \nu(v \circ j)$ for all $v \in C(j(X))$. Let us denote $y = j(r(\nu))$. Then $r(\mu) = y$. This yields $(\mu, y) \in T$. It follows from (6) that $\nu(g) = \mu(g \circ j^{-1}) = g \circ j^{-1}(y) = g(r(\nu))$.

Remark. Theorem 2 (when F contains only one element) was announced in [1] and was proved there in the special case when X is a compact convex subset of R^n.

REFERENCES

[1] Bauer, H., Leha G., Papadopoulou S., *Determination of Korovkin closures*, Math. Z. **160**, 263–274 (1979).

[2] Boboc N., Bucur Gh., *Conuri convexe de funcţii continue pe spaţii compacte*, Ed. Acad. RSR, Bucureşti 1976.

[3] Meyer P.A., *Probabilités et potentiel*, Hermann, Paris 1966.

[4] Phelps, R. R., *Lectures on Choquet's Theorem*, Van Nostrand 1966.

[5] Raşa, I., *On some results of C.A. Micchelli*, Anal. Numér. Théor., Approx. 9, 125–127 (1980).

Received 14.X.1985

Institutul Politehnic
Catedra de Matematică
3400 Cluj-Napoca, România