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ON THE HIERARCHY OF CONVEXITY OF FUNCTIONS

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In the first part of this paper we simplify the proof of the main theorem of A. M. Bruckner and E. Ostrow from [4]. In the second part we extend this result, simplifying also some proofs from our paper [8].

Let us denote the classes of continuous, convex, starshaped, respectively superadditive functions, by:

$$C(b) = \{f : [0, b] \to \mathbb{R},\ f(0) = 0,\ f \text{ continuous}\}$$

$$K(b) = \{f \in C(b);\ f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y),\ \forall\, t \in (0,1),\ \forall\, x, y \in [0,b]\}$$

$$S^*(b) = \{f \in C(b);\ f(tx) \leqslant tf(x),\ \forall\, t \in (0,1),\ x \in [0,b]\}$$

$$S(b) = \{f \in C(b);\ f(x+y) \geqslant f(x) + f(y),\ \forall x, y,\ x+y \in [0, b]\}.$$

In what follows we need some well known results (see [4]). They are more general, but we prove only the form that we use.

Lemma 1. *If the convex function f is differentiable, then f′ is non-decreasing.*

Proof. Let us suppose $x > y$. From the definition we have:

$$\frac{f(y + t(x-y)) - f(y)}{t(x-y)} \leqslant \frac{f(x) - f(y)}{x - y}$$

which gives:

$$f'(y) \leqslant \frac{f(x) - f(y)}{x - y}.$$

Replacing t by $1 - t$, we obtain similarly:

$$\frac{f(x) - f(y)}{x - y} \leqslant f'(x).$$

Lemma 2. *The function f is starshaped if and only if $f(x)/x$ is non-decreasing.*

Proof. If $0 < x < y$, from $f(ty) \leqslant tf(y)$ and $t = x/y$ we have: $f(x) \leqslant (x/y)\, f(y)$. Conversely, if $t \in (0,1)$, $tx < x$ and so $f(tx)/(tx) \leqslant f(x)/x$ gives the starshapedness of f.

Lemma 3. *If the function f is differentiable, then it is starshaped if and only if*: $f'(x) \geqslant f(x)/x$.

Proof. The function $f(x)/x$ is nondecreasing if and only if:

$$[f(x)/x]' = [f'(x)\ x - f(x)]/x^2 \geqslant 0.$$

Lemma 4. *For any* $b > 0$ *hold the inclusions*:

$$K(b) \subset S^*(b) \subset S(b).$$

Proof. a) If $f \in K(b)$, $t \in (0,1)$ and $x \in [0,b]$ then:

$$f(tx) = f(tx + (1-t)0) \leqslant tf(x) + (1-t)f(0) = tf(x)$$

that is $f \in S^*(b)$.

b) If $f \in S^*(b)$ and $x, y, x+y \in [0, b]$, then, by lemma 2, we have:

$$f(x+y) = x\,\frac{f(x+y)}{x+y} + y\,\frac{f(x+y)}{x+y} \geqslant x\,\frac{f(x)}{x} + y\,\frac{f(y)}{y}$$

and so, $f \in S(b)$.

Remark 1. These simple inclusions were not always known. So, in [5] it is proved that if f is convex and subadditive then $f(x)/x$ is non-increasing. In fact it is constant (if $f(0) = 0$).

Definition 1. *The function f has the property "P" in the mean, if the function*:

$$(1)\qquad F(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad x > 0;\ F(0) = 0$$

has the property "P".

Let us denote by: $MK(b)$, $MS^*(b)$ and $MS(b)$ the sets of functions which are convex, starshaped, respectively superadditive in the mean. The main result from [4] is:

Theorem 1. *For any* $b > 0$ *hold the strict inclusions*:

$$(2)\qquad K(b) \subset MK(b) \subset S^*(b) \subset S(b) \subset MS^*(b) \subset MS(b).$$

Proof. a) Making in (1) the change of variable: $t = xu$, it becomes (see [3]):

$$(3)\qquad F(x) = \int_0^1 f(xu)\,du.$$

If $f \in K(b)$, then for every $t \in (0,1)$ and $x, y \in [0, b]$ we have:

$$F(tx + (1-t)y) = \int_0^1 f(txu + (1-t)yu)\,du \leqslant$$

$$\leqslant \int_0^1 (t \cdot f(xu) + (1-t) \cdot f(yu))\,du = tF(x) + (1-t)F(y)$$

that is $f \in MK(b)$.

b) From (1) we have:

$$(4)\qquad f(x)/x = F'(x) + F(x)/x$$

and if F is convex F' is nondecreasing and by lemmas 4 and 2, $f \in S^*(b)$.

c) The inclusion $S^*(b) \subset S(b)$ was proved in Lemma 4. It implies also the inclusion: $MS^*(b) \subset MS(b)$.

d) Let $f \in S(b)$. Then, for every $x \in [0, b]$ and every $u \in (0,1)$:

$$f(x) = f(xu + (1-u)x) \geqslant f(xu) + f((1-u)x)$$

and so:

$$f(x) - 2F(x) = \int_0^1 (f(x) - 2\,f(xu))\,du \geqslant \int_0^1 (f((1-u)x) -$$

$$- f(xu))\,du = \int_0^1 f((1-u)x)\,du - \int_0^1 f(ux)\,du = 0.$$

But this, by Lemma 3 and by relation (4) is equivalent with $f \in MS^*(b)$.

The strictness of the inclusions (2) was proved in [3] by more examples. A beautiful proof of this fact was also given by E.F. Beckenbach in [2], showing that the function $f(x) = (1 + 1/x)\exp(-1/x)$ is in $K(1/3)$, $MK(1/2)$, $S^*((5 - 1)/2)$, $S(0.8955...)$, $MS^*(1)$ and $MS(1/\log 2)$ (the values of b being in every case the greatest possible).

Remark 2. In [6] it was considered the more general mean:

$$(5)\qquad F_g(x) = \frac{1}{g(x)}\int_0^x g'(t)f(t)\,dt,\ F_g(0) = 0.$$

Related to it, we have given in [8] the following result, whose proof we want to simplify.

Theorem 2. *If the transformation* (5) *preserves the convexity* (*the starshapedness or the superadditivity*) *then the function g is of the form*:

$$(6)\qquad g(x) = k\,x^a, \quad a > 0, \quad k \neq 0.$$

Proof. The function $f_0(x) = cx$ is in $K(b)$ for any $c \in R$, and so by lemma 4:

$$F_0(x) = \frac{c}{g(x)}\int_0^x g'(t)\ t\ dt$$

must be in $S(b)$. But c being of arbitrary sign, this happens if and only if, for $c = 1$, it verifies:

$$F_0(x+y) = F_0(x) + F_0(y)$$

for any x, y, $x + y \in [0, b]$. Thus (see [1]) : $F_0(x) = kx$ which gives (6) with $a \neq 0$. But, if $a < 0$, (5) is not defined for $f(t) = C$, thus we must take $a > 0$.

Remark 3. As was pointed out to me by prof. J.E. Pečarić, such a result was also proved by I.B. LACKOVIĆ in his doctoral dissertation using:

$$F_g(x) = \int_0^x g(t)\ f(t)\ dt \Big/ \int_0^x g(t)\ dt$$

instead of (5).

Remark 4. Denoting by F_a the function (5) with g given by (6), we have:

$$F_a(x) = \frac{a}{x^a}\int_0^x t^{a-1}\ f(t)\ dt \tag{7}$$

and so:

$$f(x) = F_a(x) + (x/a)\ F_a'(x). \tag{8}$$

If we make in (7) the substitution (see [6]) : $t = xu^{1/a}$, it becomes:

$$F_a(x) = \int_0^1 f(x\,u^{1/a})\ du. \tag{9}$$

In what follows we shall prove that the condition from theorem 2 is also sufficient. For this, let us denote by $M^aK(b)$, $M^aS^*(b)$ and $M^aS(b)$, the sets of functions $f \in C(b)$ with the property that the corresponding functions F_a belong to $K(b)$, $S^*(b)$ respectively $S(b)$.

THEOREM 3. *For any $b > 0$ and any $a > 0$ hold the following inclusions*:

$$\begin{array}{ccccccc} K(b) & \subset & M^aK(b) & \subset & S^*(b) & \subset & S(b) \\ & & & & \cap & & \cap \\ & & & & M^aS^*(b) & \subset & M^aS(b). \end{array} \tag{10}$$

Proof. a) If $f \in K(b)$, $t \in (0,1)$, $x, y \in [0, b]$, then, by (9):

$$F_a(tx + (1-t)y) = \int_0^1 f(txu^{1/a} + (1-t)\,yu^{1/a})\ du \leqslant$$

$$\leqslant \int_0^1 (tf(xu^{1/a}) + (1-t)\,f(yu^{1/a}))\ du = tF_a(x) + (1-t)\,F_a(y)$$

thus $f \in M^aK(b)$.

b) If $f \in M^aK(b)$, taking into account (8), we have:

$$f(x)/x = F_a(x)/x + F_a'(x)/a$$

thus, by lemmas 1, 2 and 4, $f \in S^*(b)$. Lemma 4 gives also the inclusions:

$$S^*(b) \subset S(b) \text{ and } M^aS^*(b) \subset M^aS(b).$$

c) If $f \in S^*(b)$, $t \in (0,1)$ and $x \in [0,b]$, using (9), we have:

$$F_a(tx) = \int_0^1 f(txu^{1/a})\ du \leqslant \int_0^1 t\,f(xu^{1/a})\ du = tF_a(x)$$

that is $f \in M^aS^*(b)$.

d) For $f \in S(b)$, x, y, $x + y \in [0, b]$, we have also:

$$F_a(x+y) = \int_0^1 f((x+y)u^{1/a})\ du \geqslant \int_0^1 (f(xu^{1/a}) + f(yu^{1/a}))\ du =$$

$$= F_a(x) + F_a(y)$$

thus $f \in M^aS(b)$.

Remark 5. To prove the strictness of the inclusions, we may poceed for $a \neq 1$ as was done in [2] for $a = 1$: let $F(x) = \exp(-1/x)$ for $x \neq 0$ and $F(0) = 0$. From (8) we get : $f(x) = (1 + 1/ax) \cdot \exp(-1/x)$ for $x \neq 0$ and $f(0) = 0$. If we denote by k, k_a, s^*, s_a^*, s, s_a the largest value of b, for what f belongs to $K(b), M^aK(b), S^*(b), M^aS^*(b), S(b)$ respectively $M^aS(b)$, we have from [2]: $k_a = 1/2$, $s_a^* = 1$ and $s_a = 1/\ln 2$. As $f''(x) \geqslant 0$ only for $x \in [(a - 4 - \sqrt{a^2+8})/(4a-4);\ (a - 4 + \sqrt{a^2+8})/(4a-4)]$, we have $k = 0$ if $0 < a < 1$ and $k = (a - 4 + \sqrt{a^2+8})/(4a-4) < 1/2$ if $a > 1$. Using Lemma 3 we have also $s^* = (a - 2 + \sqrt{a^2+4})/2a < 1$. Applying Bruckner's test (see [2]), we obtain also that s is the unique positive solution of the equation:

$$ax(\exp\ (1/x) - 2) = 4 - \exp\ (1/x)$$

thus: $1/\ln\ 4 < s < 1/\ln\ 2$. So:

$$k < k_a < s^* < s_a^* < s_a \text{ and } s < s_a.$$

We remark also that $1/\ln 4 < 1 = s_a^*$, that is, for $0 < a < 1$ we can have $s < s_a^*$ and so $S(b) \not\subset M^aS^*(b)$.

Remark 6. In [7] was proved that if $0 < a < c$ then:

$$M^aK(b) \supset M^cK(b) \text{ and } M^aS^*(b) \supset M^cS^*(b).$$

Thus (10) extends to:

$$\begin{array}{ccccccccc} K(b) & \subset & M^cK(b) & \subset & M^aK(b) & \subset & S^*(b) & \subset & S(b) \\ & & & & & & \cap & & \cap \\ & & & & & & M^cS^*(b) & \subset & M^cS(b) \\ & & & & & & \cap & & \\ & & & & & & M^aS^*(b) & \subset & M^aS(b). \end{array}$$

Moreover, if $0 < a < 1$:

$$S(b) \subset M^1S^*(b) = MS^*(b) \subset M^aS^*(b).$$

We do not know if it is true that:

$$M^cS(b) \subset M^aS(b).$$

We have proved also similar results for sequences (see [9]).

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