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THE DISTRIBUTION OF THE INDEFINITELY INCREASING ZEROS OF POLYNOMIALS WITH ALTERNATE COEFFICIENTS

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In connection with his previous paper "The asymptotic relations for the indefinitely increasing zeros of polynomials with alternate coefficients" the author gives here the proof for the inequalities

$$I_{2p,2p}(\lambda) < r_{2p+2,2p+2}(\lambda) \quad \text{and} \quad r_{2p+3,2p+3})(\lambda) < r_{2p+1,2p+1}(\lambda).$$

We consider a finite sequence of real algebraic equations with alternate coefficients

$$(1) \qquad \sum_{\nu=1}^{p}(-1)^{\nu}a_{\nu}x^{p-\nu}=0, \quad p=2,\ 3,\ 4,\ldots,n \quad (a_{\nu}>0)$$

where the coefficients $a_\nu(\nu = 1,2,3,\ldots,p)$ are constant and positive, and where $a_0 \to 0$.

Starting from the results obtained in the paper "The asymptotic relations for the indefinitely increasing zeros of polynomials with alternate coefficients" [1] we shall derive the distribution of the greatest zeros ([1], theorem (3)) $r_{p,p}(p = 2,3,4,\ldots,n)$ of equations (1) when a_0 tends to zero. This distributions is as follows

$$(2) \qquad \begin{gathered} r_{22}(a_0) < r_{44}(a_0) < \cdots < r_{2p,2p}(a_0) < r_{2p+1,2p+1}(a_0) < \cdots < r_{55}(a_0) < r_{33}(a_0), \\ a_0 < \varepsilon(p). \end{gathered}$$

The inequality $r_{2p,2p}(\lambda) < r_{2p+1,2p+1}(\lambda)$ has been proved in the mentioned paper [1] (71). Thus it remains to prove inequalities $0 < r_{2p,2p}(\lambda) < r_{2p+2,2p+2}(\lambda)$ and $0 < r_{2p+3,2p+3}(\lambda) < r_{2p+1,2p+1}(\lambda)$.

In paragraphs 1., 2. and 3. we shall prove the inequality $0 < r_{2p,2p}(\lambda) < r_{2p+2,2p+2}(\lambda)$; the inequality $0 < r_{2p+3,2p+3}(\lambda) < r_{2p+1,2p+1}(\lambda)$ will be proved in paragraph 4.

1. The angle of the tangent to the last arc of the curve $y=y_{2p+1}(x,\lambda)$ at the point of intersection of this curve and the axis X_{2p}, and the angle of the tangent to the last arc of the curve $y = y_{2p+2}(x,\lambda)$ at the point of intersection of this curve and the axis X_{2p+1}, respectively.

We consider the angle φ_{2p} which the tangent to the last arc of the curve $y = y_{2p+1}(x,\lambda)$, at the point $T_1\{r_{2p,2p}(\lambda), -a_{2p+1}\}$ of the system (X_{2p+1}), makes with the axis X_{2p}. We consider also the angle φ_{2p+1} which the tangent to the last arc of the curve $y = y_{2p+2}(x, \lambda)$, at the point $T_2\{r_{2p+1,2p+1}(\lambda), a_{2p+2}\}$ of he system (X_{2p+2}), makes with the axis X_{2p+1}. The tangent of the angle φ_{2p} is

(3) $$\operatorname{tg}\varphi_{2p} = y'_{2p+1}\{r_{2p,2p}(\lambda), \lambda\} \equiv N_{2p,2p}(\lambda).$$

From $$y_{2p+1}(x, \lambda) = x \cdot y_{2p}(x, \lambda) - a_{2p+1}$$

it follows $y'_{2p+1}(x, \lambda) = y_{2p}(x, \lambda) + x \cdot y'_{2p}(x, \lambda)$, thus

(4) $$y'_{2p+1}\{r_{2p,2p}(\lambda), \lambda\} \equiv N_{2p,2p}(\lambda) = r_{2p,2p}(\lambda) \cdot y'_{2p}\{r_{2p,2p}(\lambda)), \lambda\}.$$

In virtue of theorem [1] (4), for $\lambda = \alpha_{2p}$, the last arc of the curve $y = y_{2p}(x, \alpha_{2p})$ touches the axis X_{2p} from the side of the positive ordinates; now we have $r_{2p,2p-1}(\alpha_{2p}) \equiv r_{2p,2p}(\alpha_{2p})$. Owing to the contact of the curve $y = y_{2p}(x, \alpha_{2p})$ with the axis X_{2p} at the point $x = r_{2p,2p}(\alpha_{2p})$ we have

(5) $$y'_{2p}\{r_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} \equiv 0 \text{ and}$$

(6) $$y_{2p}\{r_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} \equiv 0.$$

From (4), owing to (5), it follows

(7) $$y'_{2p+1}\{r_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} \equiv N_{2p,2p}(\alpha_{2p}) = r_{2p,2p}(\alpha_{2p}) \cdot y'_{2p}\{r_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} \equiv 0.$$

In the previous [1] (80) we have determined

$$\lim_{\lambda \to +\infty} y'_{2p+1}\{r_{2p,2p}(\lambda), \lambda\} = +\infty; \text{ hence}$$

(8) $$\operatorname{tg}\varphi_{2p} = y'_{2p+1}\{r_{2p,2p}(\lambda), \lambda\} \equiv N_{2p,2p}(\lambda) \to +\infty, \quad \lambda \to +\infty.$$

The variable quantity $N_{2p,2p}(\lambda)$ increases from 0 to $+\infty$ when λ varies from α_{2p} to $+\infty$.

The tangent of the angle φ_{2p+1} is

(9) $$\operatorname{tg}\varphi_{2p+1} = y'_{2p+2}\{r_{2p+1,2p+1}(\lambda), \lambda\} \equiv N_{2p+1,2p+1}(\lambda).$$

From $$y_{2p+2}(x, \lambda) = x \cdot y_{2p+1}(x, \lambda) + a_{2p+2}$$

it follows $y'_{2p+2}(x, \lambda) = y_{2p+1}(x, \lambda) + x \cdot y'_{2p+1}(x, \lambda)$, thus

(10) $$y'_{2p+2}\{r_{2p+1,2p+1}(\lambda), \lambda\} \equiv N_{2p+1,2p+1}(\lambda) = r_{2p+1,2p+1}(\lambda) \cdot y'_{2p+1}\{r_{2p+1,2p+1}(\lambda), \lambda\} > 0.$$

By theorem [1] (4) the last arc of the parabola $y = y_{2p+1}(x, \lambda)$ cuts the axis X_{2p+1} at the last point of intersection $r_{2p+1,2p+1}(\lambda)$ at an acute angle; hence $y'_{2p+1}\{r_{2p+1,2p+1}(\lambda), \lambda\} > 0$ and therefore (10) is valid. The curve $y = y_{2p+1}(x, \alpha_{2p})$ passes through the point $x = r_{2p+1,2p+1}(\alpha_{2p})$ on the axis X_{2p+1} making an acute angle ψ and therefore

(11) $$y'_{2p+1}\{r_{2p+1,2p+1}(\alpha_{2p}), \alpha_{2p}\} = \operatorname{tg}\psi > 0;$$

hence

(12) $$y'_{2p+2}\{r_{2p+1,2p+1}(\alpha_{2p}), \alpha_{2p}\} \equiv N_{2p+1,2p+1}(\alpha_{2p}) > 0.$$

Thus the first derivate $y'_{2p+2}\{r_{2p+1,2p+1}(\lambda), \lambda\}$ is permanently positive for $\lambda \in [\alpha_{2p}, +\infty]$. The curve $y = y'_{2p+2}(x, \lambda)$ passes through the point $T_{2p+1}\{r_{2p+1,2p+1}(\lambda), N_{2p+1,2p+1}(\lambda)\}$ and the ordinate $N_{2p+1,2p+1}(\lambda)$ of the point T_{2p+1} is permanently positive in the system (X_{2p+1}) for each $\lambda \in [\alpha_{2p}, +\infty]$.

2. The positions of the points of intersection of the last arc of the parabola $y = y'_{2p+2}(x, \lambda)$ with the axis X_{2p} and the axis X_{2p+1} when λ varies from α_{2p} to $+\infty$.

We shall determine the expression for $y'_{2p+2}(x, \lambda)$ by means of the polynomials $y_{2p+1}(x, \lambda)$ and $y_{2p}(x, \lambda)$. From the formulae

$$y_{2p+1}(x, \lambda) = x \cdot y_{2p}(x, \lambda) - a_{2p+1}$$

and

$$y_{2p+2}(x, \lambda) = x \cdot y_{2p+1}(x, \lambda) + a_{2p+2}$$

it follows $y_{2p+2}(x, \lambda) = x^2 \cdot y_{2p}(x, \lambda) - x \cdot a_{2p+1} + a_{2p+2}$; and from this

(13) $$y'_{2p+2}(x, \lambda) = x \cdot \{2y_{2p}(x, \lambda) + x \cdot y'_{2p}(x, \lambda)\} - a_{2p+1}.$$

Whe shall transform expression (13) as follows: owing to

(14) $$y'_{2p+1}(x, \lambda) = y_{2p}(x, \lambda) - x \cdot y'_{2p}(x, \lambda)$$

from (13) we obtain

(15) $$y'_{2p+2}(x, \lambda) = x \cdot \{y_{2p}(x, \lambda) + y'_{2p+1}(x, \lambda)\} - a_{2p+1}.$$

For $x = r_{2p,2p}(\lambda)$, from (15), we obtain

(16) $$y'_{2p+2}\{r_{2p,2p}(\lambda), \lambda\} = r_{2p,2p}(\lambda) \cdot y'_{2p+1}\{r_{2p,2p}(\lambda), \lambda\} - a_{2p+1}$$

or

(17) $$y'_{2p+2}\{r_{2p,2p}(\lambda), \lambda\} = r_{2p,2p}(\lambda) \cdot N_{2p,2p}(\lambda) - a_{2p+1}, \quad \lambda \geqslant \alpha_{2p}.$$

From (17), owing to (7), it follows

(18) $$y'_{2p+2}\{r_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} = -a_{2p+1} < 0, \quad \lambda = \alpha_{2p}.$$

From $$y''_{2p+2}(x, \lambda) = 2y_{2p}(x, \lambda) + 4x \cdot y'_{2p}(x, \lambda) + x^2 \cdot y''_{2p}(x, \lambda)$$

we obtain, for $x = r_{2p,2p}(\alpha_{2p})$, the expression

(19) $$y''_{2p+2}\{r_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} = r^2_{2p,2p}(\alpha_{2p}) \cdot y''_{2p}\{r_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} > 0,$$

where $y''_{2p+2}\{r_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} > 0$ because of the minimum of the function $y = y_{2p}(x, \alpha_{2p})$ at the point $x = r_{2p,2p}(\alpha_{2p})$.

According to the aforesaid the point $T_{2p+1}\{r_{2p+1,2p+1}(\alpha_{2p}), N_{2p+1,2p+1}(\alpha_{2p})\}$ is above the axis X_{2p+1} and the point $T_1\{r_{2p,2p}(\alpha_{2p}), -a_{2p+1}\}$ is below this axis, and thereby $r_{2p,2p}(\alpha_{2p}) < r_{2p+1,2p+1}(\alpha_{2p})$. Therefore the last point of intersection $r'_{2p+2,2p+1}(\alpha_{2p})$ of the curve $y = y'_{2p+2}(x, \alpha_{2p})$ with the axis X_{2p+1}

represents the point of the last minimum fo the curve $y = y_{2p+2}(x, \alpha_{2p})$, thus

$$r'_{2p+2,2p+1}(\alpha_{2p}) \equiv {}_{2p+2}x_{\min}(\alpha_{2p})$$

and for this point there holds the following inequality

$$0 < r_{2p,2p}(\alpha_{2p}) < {}_{2p+2}x_{\min}(\alpha_{2p}) < r_{2p+1,2p+1}(\alpha_2). \tag{20}$$

By (17) and (10) the points $T_{2p}\{r_{2p,2p}(\lambda), y'_{2p+2}[r_{2p,2p}(\lambda),\lambda]\}$ and $T_{2p+1}\{r_{2p+1,2p+1}(\lambda), y'_{2p+2}[r_{2p+1,2p+1}(\lambda), \lambda]\}$ lie on the last arc of the curve $y = y'_{2p+2}(x, \lambda)$, $\lambda \geqslant \alpha_{2p}$. Expression (17) represents the system (X_{2p+1}) the ordinate of the curve $y = y'_{2p+2}(x, \lambda)$ at the point $x = r_{2p,2p}(\lambda)$. This ordinate may be $\begin{smallmatrix} > \\ = \\ < \end{smallmatrix}$ 0 depending on the magnitude of the expression $r_{2p,2p}(\lambda) \cdot N_{2p,2p}(\lambda)$. As will be seen from the following statement, the point T_{2p} lies on the axis X_{2p} for $\lambda = \alpha_{2p}$, between the axes X_{2p} and X_{2p+1} for $\lambda \in (\alpha_{2p}, \lambda^{(2p)})$, on the axis X_{2p+1} for $\lambda = \lambda^{(2p)}$ and above the axis X_{2p+1} for $\lambda > \lambda^{(2p)}$. The point T_{2p+1} lies above the axis X_{2p+1} for $\lambda \in [\alpha_{2p}, +\infty]$. Thus the last arc of the curve $y = y'_{2p+2}(x, \lambda)$ cuts the axes X_{2p} and X_{2p+1} for $\lambda > \alpha_{2p}$.

The derivate of the function $y_{2p+2}(x, \lambda)$ is

$$y'_{2p+2}(x, \lambda) = (2p+2)a_0(\lambda)x^{2p+1} - (2p+1)a_1x^{2p} + 2pa_2\, x^{2p-1} \mp \ldots - \\ -3a_{2p-1}x^2 + 2a_{2p}\, x - a_{2p+1}.$$

The curve

$$y = y'_{2p+2}(x, \lambda) \tag{21}$$

is referred to the system (X_{2p+1}). In the system (X_{2p}) the equation of the same curve (21) is

$$y = (2p+2)a_0(\lambda)x^{2p+1} - (2p+1)a_1x^{2p} + 2pa_2\, x^{2p-1} \mp \ldots - \\ - 3a_{2p+1}x^2 + 2a_{2p}\, x. \tag{22}$$

The points of intersection of curve (21) with the axis X_{2p}, i.e. with the straight line $y = -a_{2p+1}$ are obtained from the equation

$$(2p+2)a_0(\lambda)x^{2p+1} - (2p+1)a_1x^{2p} + 2pa_2x^{2p-1} \mp \ldots - 3a_{2p-1}x^2 + \\ + 2a_{2p}\, x - a_{2p+1} = -a_{2p+1},$$

namely from the equation

$$\eta_{2p}(x) \equiv (2p+2)a_0(\lambda)\, x^{2p} - (2p+1)a_1x^{2p-1} + \\ + 2pa_2x^{2p-2} \mp \ldots + 2a_{2p} = 0. \tag{23}$$

All real roots of equation (23) are positive. By $\xi(\lambda)$ we denote the greatest root of equation (23)

The derivate $y'_{2p+2}(x,\lambda)$ can be written as follows

$$y'_{2p+2}(x,\lambda) = x\{(2p+2)a_0(\lambda)x^{2p} - 2(p+1)a_1x^{2p-1} + \\ + 2pa_2x^{2p-2} \mp \ldots - 3a_{2p-1}x + 2a_{2p}\} - a_{2p+1} \equiv x \cdot \eta_{2p}(x) - a_{2p+1}. \tag{24}$$

The asymptotic expression for the function $y'_{2p+2}(x, \lambda)$ is

$$y'_{2p+2}(x, \lambda) \sim x^{2p-1}\{(2p+2)a_0(\lambda)x^2 - (2p+1)a_1x + 2pa_2\},\ x \to +\infty.$$

The equations

$$(2p+2)a_0(\lambda)x^{2p} - (2p+1)a_1x^{2p-1} + 2pa_2x^{2p-2} \mp \ldots - \\ - 3a_{2p-1}x + 2a_{2p} = 0 \tag{25}$$

and

$$(2p+2)\, a_0(\lambda)x^2 - (2p+1)a_1x + 2pa_2 = 0 \tag{26}$$

belong to a sequence in which the coefficients of their first three terms are equal, as is the case also in sequence (1). Let us denote by $r'_{2p,2p}(\lambda)$ the last root of equation (25), i.e. the last zero of the polynomial (22) and by $r'_{22}(\lambda, 2p)$ the greater root of equation (26). In virtue of the asymtotic relation [1](65) we now have the following asymptotic relation

$$\xi(\lambda) \equiv r'_{2p,2p}(\lambda) \sim r'_{22}(\lambda, 2p),\ \lambda \to +\infty, \tag{27}$$

and in virtue of [1](52) the inequality

$$0 < r'_{22}(\lambda, 2p) < r'_{2p,2p}(\lambda). \tag{28}$$

The point of intersection $r'_{2p+2,2p+1}(\lambda)$ of the last arc of the curve $y = y_{2p+2}(x, \lambda)$ with the axis X_{2p+1} represents the point of the last minimum ${}_{2p+2}x_{\min}(\lambda)$ of the curve $y = y_{2p+2}(x,\lambda)$; hence

$$r'_{2p+2,2p+1}(\lambda) \equiv {}_{2p+2}x_{\min}(\lambda),\ \ \lambda \geqslant \alpha_{2p}.$$

In connection with relation (18) we shall show that for $\lambda > \alpha_{2p}$ the ordinate of the curve $y = y'_{2p+2}(x, \lambda)$ in the system (X_{2p+1}) for the abscissa $\xi = r'_{2p,2p}(\lambda)$ has the same value $(-a_{2p+1})$, i.e. that

$$y'_{2p+2}\{r'_{2p,2p}(\lambda), \lambda\} = -a_{2p+1} \text{ when } \ \lambda > \alpha_{2p}. \tag{29}$$

In virtue of the fact that $r'_{2p,2p}(\lambda)$ is the last zero of the polynomial $\eta_{2p}(x)$, from (24) it follows (29). From

$$y'_{2p+2}(x, \lambda) + a_{2p+1} = x \cdot \eta_{2p}(x), \tag{30}$$

in virtue of relation (13), it follows

$$\eta_{2p}(x) = 2y_{2p}(x, \lambda) + x \cdot y'_{2p}(x, \lambda) \tag{31}$$

and, by (5) and (6), from this it follows

$$\eta_{2p}\{r'_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} \equiv 0, \quad \text{hence} \tag{32}$$

$$\xi(\alpha_{2p}) \equiv r'_{2p,2p}(\alpha_{2p}) = r_{2p,2p}(\alpha_{2p}).$$

Since $r_{2p,2p}(\lambda)$ is the zero of the polynomial $y_{2p}(x, \lambda)$, from (31) it follows

$$\eta_{2p}\{r_{2p,2p}(\lambda)\} = r_{2p,2p}(\lambda) \cdot y'_{2p}\{r_{2p,2p}(\lambda), \lambda\} \equiv N_{2p,2p}(\lambda) > 0,\quad \lambda > \alpha_{2p}. \tag{33}$$

By theorem [1] (4) the last arc of the curve $y = y'_{2p+2}(x, \lambda)$ for $\lambda > \alpha_{2p}$ has at the point $x = r''_{2p+2,2p}(\lambda)$ a negative minimum of an arbitrarily great absolute value, so that the ordinate $y'_{2p+2}\{r''_{2p+2,2p}(\lambda), \lambda\}$ intersects the axis X_{2p} lying below the axis X_{2p+1}. This arc is concave to the right of the straight line $x = r''_{2p+2,2p}(\lambda)$. Therefore the last point of intersection $\xi(\lambda) = r'_{2p,2p}(\lambda)$ of the curve $y = y'_{2p+2}(x, \lambda)$ with the laxis X_{2p} lies to the right of the straight line $x = r''_{2p+2,2p}(\lambda)$. Since the last arc of the curve $y = y'_{2p+2}(x, \lambda)$ increases monotonously in the interval $(r''_{2p+2,2p}(\lambda), +\infty)$, there exists the inequality

(34) $$0 < r''_{2p+2,2p}(\lambda) < r'_{2p,2p}(\lambda).$$

We shall determine the ordinate of the curve (30) for $x = r_{2p,2p}(\lambda)$, $\lambda > \alpha_{2p}$. This ordinate is referred to the system (X_{2p}) since also curve (30) itself and curve (22) respectively are referred to this system. In virtue of (33) the ordinate of curves (30) and (22) respectively, for $x = r_{2p,2p}(\lambda)$, is

(35) $$y = r_{2p,2p}(\lambda) \cdot \eta_{2p}\{r_{2p,2p}(\lambda)\} = r_{2p,2p}(\lambda) \cdot N_{2p,2p}(\lambda) > 0, \quad \lambda > \alpha_{2p}.$$

Consequently ordinate (35) of the curve (30) is positive at the point $x = r_{2p,2p}(\lambda)$, for $\lambda > \alpha_{2p}$; and for $\lambda = \alpha_{2p}$, owing to (7), this ordinate is equal to zero. Since curve (30) increases monotonously in the interval $(r'_{2p,2p}(\lambda), +\infty)$, we have, owing to (35), the following inequality

(36) $$0 < r'_{2p,2p}(\lambda) < r_{2p,2p}(\lambda), \quad \lambda > \alpha_{2p},$$

thus, in virtue of $0 < r_{2p,2p}(\lambda) < r_{2p+1,2p+1}(\lambda)$, $\lambda > \alpha_{2p}$ [1] (71), there is

(37) $$0 < r'_{2p,2p}(\lambda) < r_{2p,2p}(\lambda) < r_{2p+1,2p+1}(\lambda), \quad \lambda > \alpha_{2p}.$$

Owing to (11) the ordinate of the curve $y = y_{2p+2}(x, \lambda)$ in the system (X_{2p+1}) is positive for $x = r_{2p+1,2p+1}(\lambda)$, and owing to (29) the ordinate of this curve in the same system is negative for $x = r'_{2p,2p}(\lambda)$. Therefore the point of intersection of the last arc of the curve $y = y'_{2p+2}(x, \lambda)$ with the axis X_{2p+1}, i.e. the point of the last minimum ${}_{2p+2}x_{\min}(\lambda)$ of the curve $y = y_{2p+2}(x, \lambda)$ is situated in the interval

(38) $$0 < r'_{2p,2p}(\lambda) < {}_{2p+2}x_{\min}(\lambda) < r_{2p+1,2p+1}(\lambda), \quad \lambda > \alpha_{2p}.$$

With regard to inequalities (37) and (38) we shall show that for a certain $\lambda^{(2p)} > \alpha_{2p}$ there will be

(39) $${}_{2p+2}x_{\min}(\lambda^{(2p)}) = r_{2p,2p}(\lambda^{(2p)}).$$

Let us determine the point of intersection of the straight line $x = r_{2p,2p}(\lambda)$, $\lambda \geqslant \alpha_{2p}$ and the curve $y = y'_{2p+2}(x, \lambda)$, that is, the ordinate of the curve $y = y'_{2p+2}(x, \lambda)$ in the system (X_{2p+1}) for the abscissa $x = r_{2p,2p}(\lambda)$, $\lambda \geqslant \alpha_{2p}$. This ordinate has been given by formula (17).

Since expression (35) is possitive, expression (17) may be $\gtreqless 0$. This depends, in the case of λ being constant, on the magnitude of the number $a_{2p+1} > 0$. For $\lambda = \alpha_{2p}$ we have $N_{2p,2p}(\alpha_{2p}) = 0$ and therefore from (17) we obtain (18). This means that the last arc of the curve $y = y'_{2p+2}(x, \alpha_{2p})$ passes through the point $(r_{2p,2p}(\alpha_{2p}), -a_{2p+1})$ on the axis X_{2p}. This arc is increasing and concave to the right of the straight line $x = r_{2p,2p}(\alpha_{2p})$; hence the point of intersection ${}_{2p+2}x_{\min}(\alpha_{2p})$ of this arc with the axis X_{2p+1} lies to the right of the point $r_{2p,2p}(\alpha_{2p})$,

(40) $$0 < r'_{2p,2p}(\alpha_{2p}) \equiv r_{2p,2p}(\alpha_{2p}) < {}_{2p+2}x_{\min}(\alpha_{2p}).$$

When λ increases from α_{2p} to $+\infty$, the value of the first derivate $y'_{2p+2}\{r_{2p,2p}(\lambda), \lambda\}$, as the value of a polynomial of odd degree, varies in the interval $(-a_{2p+1}, +\infty)$, beginning with the value $(-a_{2p+1})$ for $\lambda = \alpha_{2p}$ and ending with the value $+\infty$ for $\lambda = +\infty$, since when λ tends to $+\infty$ then $r_{2p,2p}(\lambda)$ also tends to $+\infty$, and therefore $y'_{2p+2}\{r_{2p,2p}(\lambda), \lambda\} \to +\infty$. Hence for a certain $\lambda = \lambda^{(2p)}$ there will be

(41) $$y'_{2p+2}\{r_{2p,2p}(\lambda^{(2p)}), \lambda^{(2p)}\} = r_{2p,2p}(\lambda^{(2p)}) \cdot N_{2p,2p}(\lambda^{(2p)}) - a_{2p+1}) \equiv 0.$$

Owing to (18) and (41) we have

(42) $$y'_{2p+2}\{r_{2p,2p}(\lambda), \lambda\} = r_{2p,2p}(\lambda) \cdot N_{2p,2p}(\lambda) - a_{2p+1} < 0,$$

for $\alpha_{2p} \leqslant \lambda < \lambda^{(2p)}$,

(43) $$y'_{2p+2}\{r_{2p,2p}(\lambda), \lambda\} = r_{2p,2p}(\lambda) \cdot N_{2p,2p}(\lambda) - a_{2p+1} > 0,$$

for $\lambda > \lambda^{(2p)}$.

With regard to inequalities (37) and (38) we shall investigate the mutual position of the points $r_{2p,2p}(\lambda)$ and ${}_{2p+2}x_{\min}(\lambda)$ when λ varies from α_{2p} to $+\infty$.

For $\lambda = \alpha_{2p}$ there exists inequality (40). In this case, by (18), the straight line $x = r_{2p,2p}(\alpha_{2p})$ intersects the curve $y = y'_{2p+2}(x, \alpha_{2p})$ exactly on the axis X_{2p} at the point $(r_{2p,2p}(\alpha_{2p}), -a_{2p+1})$.

For $\alpha_{2p} < \lambda < \lambda^{(2p)}$ relation (42) holds, i.e. the ordinate of the point of intersection of the straight line $x = r_{2p,2p}(\lambda)$ and the curve $y = y'_{2p+2}(x, \lambda)$ is negative in the system (X_{2p+1}), but this ordinate is, in absolute value, less than a_{2p+1}; since this curve is increasing in the interval $(r'_{2p,2p}(\lambda), +\infty)$, there exists the inequality

(44) $$0 < r'_{2p,2p}(\lambda) < r_{2p,2p}(\lambda) < {}_{2p+2}x_{\min}(\lambda), \quad \lambda \in (\alpha_{2p}, \lambda^{(2p)}).$$

For $\lambda = \lambda^{(2p)}$, by (41), the straight line $x = r_{2p,2p}(\lambda^{(2p)})$ intersects the curve $y = y'_{2p+2}(x, \lambda^{(2p)})$ exactly on the axis X_{2p+1} and now there exists the relation

(45) $$0 < r'_{2p,2p}(\lambda^{(2p)}) < r_{2p,2p}(\lambda^{(2p)}) \equiv {}_{2p+2}x_{\min}(\lambda^{(2p)}).$$

For $\lambda > \lambda^{(2p)}$, by (43), the straight line $x = r_{2p,2p}(\lambda)$ intersects the curve $y = y'_{2p+2}(x, \lambda)$ above the axis X_{2p+1}. Now there exists the inequality

(46) $$0 < r'_{2p,2p}(\lambda) < {}_{2p+2}x_{\min}(\lambda) < r_{2p,2p}(\lambda), \quad \lambda > \lambda^{(2p)};$$

thus the point ${}_{2p+2}x_{\min}(\lambda)$ remains finally on the left of the point $r_{2p,2p}(\lambda)$ for $\lambda > \lambda^{(2p)}$.

The right branch of the last arc of the parabola $y = y'_{2p+2}(x, \lambda)$ passing through the points $\{r'_{2p,2p}(\lambda), -a_{2p+1}\}$, $\{{}_{2p+2}x_{\min}(\lambda), 0\}$ and $\{r_{2p,2p}(\lambda), y'_{2p+2}[r_{2p,2p}(\lambda), \lambda]\}$ in the system (X_{2p+1}) intersects the axis X_{2p} at the point $r'_{2p+2p}(\lambda)$ at an acute angle θ_{2p}; the same arc intersects the axis X_{2p+1} at the point ${}_{2p+2}x_{\min}(\lambda)$ at an acute angle θ_{2p+1}. The axis X_{2p+1} is above the axis X_{2p} and the point ${}_{2p+2}x_{\min}(\lambda)$ is to the right of the point $r'_{2p,2p}(\lambda)$ for $\lambda > \alpha_{2p}$. Owing to the concavity of the mentioned arc and owing to the positions of the mentioned points $\theta_{2p+1} > \theta_{2p}$; hence we have

(47) $\operatorname{tg}\theta_{2p} < \operatorname{tg}\theta_{2p+1}$ and

(48) $\operatorname{tg}\theta_{2p} = y''_{2p+2}\{r'_{2p,2p}(\lambda), \lambda\}.$

In virtue of [1] (21) from (28) it follows

(49) $r'_{2p,2p}(\lambda) \to +\infty, \quad \lambda \to +\infty,$ therefore

(50) $\operatorname{tg}\theta_{2p} = y''_{2p+2}\{r'_{2p,2p}(\lambda), \lambda\} \to +\infty, \quad \lambda \to +\infty$

and owing to (47) also

(51) $\operatorname{tg}\theta_{2p+1} \to +\infty, \quad \lambda \to +\infty.$

From formulae (50) and (51) it follows that the two angles θ_{2p} and θ_{2p+1} tend to the common limiting value $\frac{\pi}{2}$ as $\lambda \to +\infty$. This means that the points of intersection $r'_{2p,2p}(\lambda)$ and ${}_{2p+2}x_{\min}(\lambda)$ with the axis X_{2p} and the axis X_{2p+1} respectively have abscissas approaching more and more closely each other when λ increases, i.e. that we have

(52) $r'_{2p,2p}(\lambda) \sim {}_{2p+2}x_{\min}(\lambda), \quad \lambda \to +\infty$

and that, by (44), (45) and (46), for each $\lambda > \alpha_{2p}$ there exists the inequality

(53) $0 < r'_{2p,2p}(\lambda) < {}_{2p+2}x_{\min}(\lambda).$

From the asymptotic relation [1] (27) it follows

$$\{r_{22}(\lambda) - r'_{22}(\lambda, 2p)\} \to +\infty, \quad \lambda \to +\infty$$

and owing to (27) and [1] (65) we obtain

$$\{r_{2p,2p}(\lambda) - r'_{2p,2p}(\lambda)\} \to +\infty, \quad \lambda \to +\infty$$

and from this, owing to (52),

(54) $\{r_{2p,2p}(\lambda) - {}_{2p+2}x_{\min}(\lambda)\} \to +\infty, \quad \lambda \to +\infty.$

From relations (52) and (54) it follows that in inequality (46) the first two points approach indefinitely each other and that the last two points of this inequality recede indefinitely from each other when $\lambda > \lambda^{(2p)}$ increases indefinitely, i.e. the point $r'_{2p,2p}(\lambda)$ approaches from the left side the point ${}_{2p+2}x_{\min}(\lambda)$, while the point $r_{2p,2p}(\lambda)$ recedes indefinitely to the right from the point ${}_{2p+2}x_{\min}(\lambda)$.

The inflectional tangent of the curve $y = y_{2p+2}(x, \lambda)$ at the last point of inflection $r''_{2p+2,2p}(\lambda)$ makes with the axis X_{2p+2} an obtuse angle $\left(\frac{\pi}{2} + \varepsilon\right)$,

$\varepsilon > 0$, where $\varepsilon \to 0$ when $\lambda \to +\infty$, since by [1] (12) and [1] (37), from

$${}_{2p+2}y'_2\{r''_{22}(\lambda, 2p), \lambda\} = \{r''_{22}(\lambda, 2p)\}^{2p-1} \cdot \eta_2\{r''_{22}(\lambda, 2p), 2p\} \quad \text{and}$$

$$\eta_2\{r''_{22}(\lambda, 2p), 2p\} = \frac{-2a_2p}{p+1}\left\{\lambda + \sqrt{\lambda^2 - \frac{(2p-1)(2p+2)}{2p(2p+1)}} \cdot \lambda - \frac{2p+2}{2p+1}\right\} \to -\infty, \quad \lambda \to +\infty$$

it follows ${}_{2p+2}y'_2\{r''_{22}(\lambda, 2p), \lambda\} \to -\infty, \ \lambda \to +\infty$ and by [1] (23), [1] (61) and [1] (67) we obtain $r''_{22}(\lambda, 2p) \gtrsim r''_{2p+2,2p}(\lambda) \to +\infty, \quad \lambda \to +\infty$

hence $y'_{2p+2}\{r''_{2p+2,2p}(\lambda), \lambda\} \to -\infty, \quad \lambda \to +\infty.$

For $x = r'_{2p,2p}(\lambda)$ the tangent to the curve $y = y_{2p+2}(x,\lambda)$ has a constant slope $(-a_{2p+1})$, and for $x = r_{2p,2p}(\lambda)$ the tangent to the same curve has a variable slope (42).

Owing to inequalities (44), (45) and (46) and owing to asymptotic relation (52) the points of contact of these two tangents approach, along the left branch of the last arc, the point of minimum for $x = {}_{2p+2}x_{\min}(\lambda)$, while $\lambda < \lambda^{(2p)}$. If $\lambda = \lambda^{(2p)}$, then the tangent at the point of contact is horizontal for $x = r_{2p,2p}(\lambda^{(2p)})$ and the point of contact of the tangent, for $x = r'_{2p,2p}(\lambda^{(2p)})$, is to the left of the straight line $x = r_{2p,2p}(\lambda^{(2p)}) \equiv {}_{2p+2}x_{\min}\lambda^{(2p)})$. If $\lambda > \lambda^{(2p)}$, the point of contact of the tangent, for $x = r'_{2p,2p}(\lambda)$, approaches, again along the left branch of the last arc, arbitrarily closely the point of minimum for $x = {}_{2p+2}x_{\min}(\lambda)$, which means that the curve $y = y_{2p+2}(x, \lambda)$ changes very rapidly but continuously the slope from $(-a_{2p+1})$ to O in a very small interval $(r'_{2p,2p}(\lambda), {}_{2p+2}x_{\min}(\lambda))$. If $\lambda > \lambda^{(2p)}$, the point of contact of the tangent, for $x = r_{2p,2p}(\lambda)$, lies on the right branch of the last arc and the slope (43) of this tangent is positive and increases indefinitely with λ.

3. The mutual position of the zeros $r_{2p,2p}(\lambda)$ and $r_{2p+2,2p+2}(\lambda)$ when λ increases indefinitely.

We consider the position of the zero $r_{2p+2,2p+2}(\lambda)$ of the polynomial $y_{2p+2}(x, \lambda)$ in relation to the position of the zero $r_{2p,2p}(\lambda)$ of the polynomial $y_{2p}(x, \lambda)$ when λ tends to $+\infty$. In this connection it is necessary to make a remark concerning the last arc of the parabola.

Therefore we consider the concave arc of a curve, not necessarily of a parabola, $y = f(x) < 0$ on the segment $[a, b]$, $0 < a < b$; the functions $f(x)$, $f'(x)$ and $f''(x)$ are continuous on this segment; $f(a) = -A < 0$, $f(b) = 0$, $f'(a) < 0$, $f'(b) > 0$, $f'(x_0) = 0, a < x_0 < b$, $f(x_0) = -M_0 = \mathrm{Min}$; $f''(x) < 0$ for $x \in [a, b]$. Let the point $(a, -A)$ be the point of inflection of the curve $y = f(x)$, whereby $x = b$ is the first zero behind the point $x = a$.

$$g_1(x) = x \cdot f(x), \quad g'_1(x_0) = f(x_0) + x_0 \cdot f'(x_0) = f(x_0) = -M_0.$$

The curve $y = g_1(x) = x \cdot f(x)$ is decreasing at the point $x = x_0$, i.e. the abscissa of the minimum x_1 of the curve $y = x \cdot f(x)$ is to the right of the point x_0; $a < x_0 < x_1 < b$.

Hence $\quad g_1(x_1) < g_1(x_0) = x_0 \cdot f(x_0) = -x_0 M_0$, i.e.

(55) $\quad g_1(x_1) \equiv -M_1 < M_1 < -x_0 M_0, \quad M_1 > x_0 M_0.$

For $g_2(x) = x \cdot g_1(x) = x^2 \cdot f(x)$ we obtain in the same manner

$$g_2'(x_1) = g_1(x_1) = x_1 \cdot f(x_1) = -M_1.$$

The curve $y = g_2(x) = x \cdot g_1(x)$ is decreasing at the point $x = x_1$, i.e. the abscissa of the minimum x_2 of the curve $y = x^2 \cdot f(x)$ is to the right of the point $x = x_1$; $a < x_0 < x_1 < x_2 < b$.

Hence $\quad g_2(x_2) < g_2(x_1) = x_1 \cdot g_1(x_1) = -x_1 M_1,$

i.e. $\quad g_2(x_2) \equiv -M_2 < -x_1 M_1,$

(56) $\quad M_2 > x_1 M_1 > x_1 \cdot x_0 M_0, \quad M_2 > x_1 x_0 M_0.$

For $y = g_n(x) = x \cdot g_{n-1}(x) = x^n \cdot f(x), \quad n = 1,2,3,\ldots$ we obtain

(57) $\quad M_n > M_0 \cdot \prod_{\nu=0}^{n-1} x_\nu, \quad g_n'(a) = -na^{n-1} \cdot A - a^n \cdot |f'(a)| < 0, \quad g_n'(b) = b^n \cdot f(b) > 0,$

(58) $\quad g_n'(x_{n-1}) = g_{n-1}(x_{n-1}) = x_{n-1}^{n-1} \cdot f(x_{n-1}) = -M_{n-1}, \; a < x_0 < x_n < b.$

By inequality (58) we have

(59) $\quad {}_{2p}x_{\min}(\lambda) < {}_{2p+1}x_{\min}(\lambda) < {}_{2p+2}x_{\min}(\lambda), \quad \lambda > \alpha_{2p}.$

In virtue of theorem [1] (6), for $\lambda < \alpha_{2p}$, the real zeros $r_{2p,2p-1}(\lambda)$ and $r_{2p,2p}(\lambda)$ of the polynomial $y_{2p}(x, \lambda)$ do not exist; instead of them we have two conjugate complex zeros of the polynomial $y_{2p}(x, \lambda)$. The curve $y = y_{2p}(x, \alpha_{2p})$ touches the axis X_{2p} from the side of the positive ordinates of the system (X_{2p}) at the point $x = r_{2p,2p}(\alpha_{2p})$ and for this reason there exist relations (5), (6) and (18). The curve

(60) $\quad y = {}_{2p+2}y_{2p+1}(x, \alpha_{2p}) \equiv x \cdot y_{2p+1}(x, a_{2p})$

is referred to the system (X_{2p+1}). This curve passes through the point $r_{2p+1,2p+1}(\alpha_{2p})$ on the axis X_{2p+1}. From $y'_{2p+2}(x, \alpha_{2p}) = {}_{2p+2}y_{2p+1}(x, \alpha_{2p})$ and in virtue of (18) it follows

$$ {}_{2p+2}y'_{2p+1}\{r_{2p,2p}(\alpha_{2p}), \alpha_{2p}\} = -a_{2p+1};$$

hence the curve $y = {}_{2p+2}y_{2p+1}(x, \alpha_{2p})$ decreases at the point whose abscissa is $r_{2p,2p}(\alpha_{2p})$, and from this it follows that the abscissa of the last minimum ${}_{2p+2}x_{\min}(\alpha_{2p})$ of the curve $y = {}_{2p+2}y_{2p+1}(x, \alpha_{2p})$ is situated in the interval $(r_{2p,2p}(\alpha_{2p}), r_{2p+1,2p+1}(\alpha_{2p}))$. By ${}_{2p+2}M_{2p+1}(\alpha_{2p})$ we denote the value of the last minimum of curve (60).

For the absolute value $|{}_{2p+2}M_{2p+1}(\alpha_{2p})$ and for the coefficient a_{2p+2} there exist the following possibilities:

1° If $a_{2p+2} < |{}_{2p+2}M_{2p+1}(\alpha_{2p})|$, then the zero $r_{2p+2,2p+2}(\alpha_{2p})$ is to the right of ${}_{2p+2}x_{\min}(\alpha_{2p})$ and to the left of $r_{2p+1,2p+1}(\alpha_{2p})$ since the axis X_{2p+2} is below the axis X_{2p+1}; and owing to (40) we have the following distribution

$$0 < r_{2p,2p}(\alpha_{2p}) < {}_{2p+2}x_{\min}(\alpha_{2p}) < r_{2p+2,2p+2}(\alpha_{2p}) < r_{2p+1,2p+1}(\alpha_{2p}) \quad \text{or}$$

(61) $\quad 0 < r_{2p,2p}(\alpha_{2p}) < r_{2p+2,2p+2}(\alpha_{2p}) < r_{2p+1,2p+1}(\alpha_{2p}).$

2° If $a_{2p+2} = |{}_{2p+2}M_{2p+1}(\alpha_{2p})|$, then there exist the equalities $r_{2p+2,2p+1}(\alpha_{2p}) = r_{2p+2,2p+2}(\alpha_{2p}) = {}_{2p+2}x_{\min}(\alpha_{2p})$ and owing to (40) we have again (61).

3° If $a_{2p+2p+2} > |{}_{2p+2}M_{2p+1}(\alpha_{2p})|$, then instead of the real zeros $r_{2p+2,2p+1}(\alpha_{2p})$ and $r_{2p+2,2p+2}(\alpha_{2p})$ we have, by theorem [1] (6), two conjugate complex zeros.

For $\lambda_2 > \lambda_1$ the curve $y = {}_{2p+2}y_{2p+1}(x, \lambda_2)$ is below the curve $y = {}_{2p+2}y_{2p+1}(x, \lambda_1)$ since $a_0(\lambda_1) > a_2(\lambda_2)$; thus

(62) $$ {}_{2p+2}y_{2p+1}(x, \lambda_1) - {}_{2p+2}y_{2p+1}(x, \lambda_2) = [a_0(\lambda_1) - a_0(\lambda_2)] \cdot x^{2p+2} > 0, \quad \text{for } x \neq 0.$$

Owing to $\lambda^{(2p)} > \alpha_{2p}$ the curve $y = {}_{2p+2}y_{2p+1}(x, \lambda^{(2p)})$ is below the curve $y = {}_{2p+2}y_{2p+1}(x, \alpha_{2p})$ for each $x \neq 0$, thus also for $x = {}_{2p+2}x_{\min}(\alpha_{2p})$. Hence we have

(63) $\quad |{}_{2p+2}M_{2p+1}(\alpha_{2p})| < |{}_{2p+2}M_{2p+1}(\lambda^{(2p)})|.$

In order to investigate the positions of the zeros $r_{2p,2p}(\lambda)$, $r_{2p+2,2p+2}(\lambda)$ and $r_{2p+1,2p+1}(\lambda)$ the following three cases are to be distinguished:

a) $a_{2p+2} < |{}_{2p+2}M_{2p+1}(\lambda^{(2p)})|$, b) $a_{2p+2} = |{}_{2p+2}M_{2p+1}(\lambda^{(2p)})|$,

c) $a_{2p+2} > |{}_{2p+2}M_{2p+1}(\lambda^{(2)})|$.

We shall consider only the case c) since then, as we shall show, the point $r_{2p,2p}(\lambda)$ remains finally on the left of the point $r_{2p+2,2p+2}(\lambda)$, for $\lambda > \bar{\lambda} > \alpha_{2p+2} > \lambda^{(2p)} > \alpha_{2p}$.

We consider at once the case when the axis X_{2p+2} is situated below the axis X_{2p}, i.e. when $a_{2p+2} > |{}_{2p+2}M_{2p+1}(\lambda^{(2)})| > a_{2p+1}$. By theorem [1] (5) the last arc of the parabola $y = y_{2p+2}(x, \alpha_{2p+2})$ touches the axis X_{2p+2} only for a certain $\alpha_{2p+2} > \lambda^{(2p)}$. Now there is

(64) $\quad r_{2p+2,2p+1}(\alpha_{2p+2}) \equiv r_{2p+2,2p+2}(\alpha_{2p+2}) \equiv {}_{2p+2}x_{\min}(\alpha_{2p+2});$

and since for $\alpha_{2p+2} > \lambda^{(2p)}$, by (46), there holds

(65) $\quad 0 < {}_{2p+2}x_{\min}(\alpha_{2p+2}) < r_{2p,2p}(\alpha_{2p+2}).$

from (64) and (65) it follows

(66) $\quad 0 < r_{2p+2,2p+2}(\alpha_{2p+2}) < r_{2p,2p}(\alpha_{2p+2}).$

In order for the right branch of the last arc of the curve $y = y_{2p+2}(x, \lambda)$ to cut the axis X_{2p+2} there should now be $\lambda > \alpha_{2p+2} > \lambda^{(2p)}$ and therefore inequalities (43) and (46) are valid. We shall calculate the ordinate of the curve $y = {}_{2p+2}y_{2p+1}(x, \lambda)$ for $x = r_{2p+2}(\lambda)$ in the system (X_{2p+1})

$$(67)\quad {}_{2p+2}y_{2p+1}\{r_{2p,2p}(\lambda), \lambda\} = r^2_{2p,2p}(\lambda)\ [a_0(\lambda)\ r^{2p}_{2p,2p}(\lambda) - a_1 r^{2p-1}_{2p,2p}(\lambda) \pm$$

$$\pm \dots + a_{2p}] - a_{2p+1} r_{2p,2p}(\lambda) = -a_{2p+1} \cdot r_{2p,2p}(\lambda) < 0,\quad \lambda > \alpha_{2p+2} > \lambda^{(2p)}.$$

The absolute value of this ordinate is

$$(68)\quad |{}_{2p+2}y_{2p+1}\{r_{2p,2p}(\lambda), \lambda\}| = |-a_{2p+1} \cdot r_{2p,2p}(\lambda)| = a_{2p+1} \cdot r_{2p,2p}(\lambda),$$

$$\lambda > \alpha_{2p+2} > \lambda^{(2p)}.$$

If we denote the extreme point of this ordinate by $P(\lambda)$ and the projection of this point on the axis X_{2p+1} by $P'(\lambda)$ and the absolute value of this ordinate by $\overline{PP'}(\lambda)$ then we have

$$(69)\quad \overline{PP'}(\lambda) = a_{2p+1} \cdot r_{2p,2p}(\lambda),\quad \lambda > \alpha_{2p+2} > \lambda^{(2p)}.$$

From (69) it follows that $\overline{PP'}(\lambda)$ increases with λ since $r_{2p,2p}(\lambda)$ increases with λ. For this reason there exists a certain $\overline{\lambda} > \alpha_{2p+2} > \lambda^{(2p)}$ for which there is

$$(70)\quad a_{2p+2} = \overline{PP'}(\overline{\lambda}) = a_{2p+1} \cdot r_{2p,2p}(\overline{\lambda}),\quad \overline{\lambda} > \alpha_{2p+2} > \lambda^{(2p)}.$$

For the curve $y = y_{2p+2}(x, \overline{\lambda})$ the point $P(\overline{\lambda})$ lies exactly on the axis X_{2p+2}. Thus the curve $y = y_{2p+2}(x, \overline{\lambda})$ cuts the axis X_{2p+2} at the point

$$r_{2p,2p}(\overline{\lambda}) = r_{2p+2,2p+2}(\overline{\lambda}),\quad \overline{\lambda} > \alpha_{2p+2} > \lambda^{(2p)},$$

hence, by (46), we have

$$(71)\quad 0 < {}_{2p+2}x_{\min}(\overline{\lambda}) < r_{2p,2p}(\overline{\lambda}) \equiv r_{2p+2,2p+2}(\overline{\lambda}),\quad \overline{\lambda} > \alpha_{2p+2} > \lambda^{(2p)}.$$

It should be mentioned that the point of the last minimum of the curve $y = y_{2p+2}(x, \overline{\lambda})$ lies below the axis X_{2p+2}. For $\lambda > \overline{\lambda} > \alpha_{2p+2}$ we have

$$(72)\quad \overline{PP'}(\lambda) = a_{2p+1} \cdot r_{2p,2p}(\lambda) > a_{2p+2},\quad \lambda > \overline{\lambda} > \alpha_{2p+2} > \lambda^{(2p)}.$$

Now the point $P(\lambda)$ is below the axis X_{2p+2} just as the point of the last minimum of the same curve. Owing to (43) the curve $y = y_{2p+2}(x, \lambda)$ is increasing to the right of the straight line $x = r_{2p,2p}(\lambda)$; therefore the point of intersection $r_{2p+2,2p+2}(\lambda)$ of this curve and the axis X_{2p+2} is situated to the right of the straight line $x = r_{2p,2p}(\lambda)$; hence

$$(73)\quad 0 < r_{2p,2p}(\lambda) < r_{2p+2,2p+2}(\lambda),\quad \lambda > \overline{\lambda} > \alpha_{2p+2} > \lambda^{(2p)}$$

and, since the axis X_{2p+1} is above the axis X_{2p}, there exists the inequality

$$(74)\quad 0 < r_{2p,2p}(\lambda) < r_{2p+2,2p+2}(\lambda) < r_{2p+1,2p+1}(\lambda)$$

for each $\lambda > \overline{\lambda} > \alpha_{2p+2} > \lambda^{(2p)}$ and for each $a_{2p+2} > |{}_{2p+2}M_{2p+1}(\lambda^{(2p)})|$.

For all values of the parameter λ from the interval $(\alpha_{2p+2}, \overline{\lambda})$ the point of the last minimum of the curve $y = y_{2p+2}(x, \lambda)$ lies below the axis X_{2p+2}, but at the same time we have

$$\overline{PP'}(\lambda) < a_{2p+2},\quad \lambda \in (\alpha_{2p+2}, \overline{\lambda}).$$

Hence the last point of intersection of the curve $y = y_{2p+2}(x, \lambda)$, $\lambda \in (\alpha_{2p+2}, \overline{\lambda})$, and the axis X_{2p+2} lies to the left of the straight line $x = r_{2p,2p}(\lambda)$. Therefore we have

$$(75)\quad 0 < r_{2p+2,2p+2}(\lambda) < r_{2p,2p}(\lambda)\ \text{ for }\ \lambda \in (\alpha_{2p+2}, \overline{\lambda}).$$

4. The proof of the inequality $r_{2p+3,2p+3}(\lambda) < r_{2p+1,2p+1}(\lambda)$.

By theorem [1] (5) one of the curves of the family

$$(76)\quad y = y_{2p+2}(x, \lambda) = a_0(\lambda) x^{2p+2} - a_1 x^{2p+1} + a_2 x^{2p} \mp \dots + (-1)^{2p+1} a_{2p+1} x + a_{2p+2},$$

determined by a special value $\lambda = \alpha_{2p+2}$ of the parameter λ, touches the axis X_{2p+2} from the side of the positive ordinates at the point of its last minimum. This point of contact is $r_{2p+2,2p+1}(\alpha_{2p+2}) \equiv r_{2p+2,2p+2}(\alpha_{2p+2})$. Depending on the coefficients of the polynomial $y_{2p+2}(x, \alpha_{2p+2})$ there may be $r_{2p+2,2p+2}(\alpha_{2p+2}) \lesseqgtr 1$.

For $\lambda > \alpha_{2p+2}$ the right branch of the last arc of parabola (76) cuts the axis X_{2p+2} at the point $r_{2p+2,2p+2}(\lambda)$ and, by [1] (53), the value of this root increases indefinitely with λ. If $r_{2p+2,2p+2}(\alpha_{2p+2}) < 1$, then there will be $r_{2p+2,2p+2}(\lambda_0) = 1$ for a certain value $\lambda_0 > \alpha_{2p+2}$; if $r_{2p+2,2p+2}(\alpha_{2p+2}) \geqslant 1$, then there is $\lambda_0 = \alpha_{2p+2}$. Hence

$$(77)\quad x \cdot y_{2p+2}(x, \lambda_0) > y_{2p+2}(x, \lambda_0)\ \text{ for }\ x > r_{2p+2,2p+2}(\lambda_0).$$

From this it follows that the right branch of the last arc of the parabola

$$(78)\quad y = {}_{2p+3}y_{2p+2}(x, \lambda_0) \equiv x \cdot y_{2p+2}(x, \lambda_0),\quad \lambda_0 \geqslant \alpha_{2p+2}$$

passes through the same point $x = r_{2p+2,2p+2}(\lambda_0)$ of the axis X_{2p+2} and that this branch lies above the right branch of the last arc of the parabola $y = y_{2p+2}(x, \lambda_0)$ for $x > r_{2p+2,2p+2}(\lambda_0)$.

The right branch of the last arc of the parabola

$$(79)\quad y = y_{2p+2}(x, \lambda_0) = x \cdot y_{2p+1}(x, \lambda_0) + a_{2p+2},\quad \lambda_0 \geqslant \alpha_{2p+2}$$

cuts the axis X_{2p+2} at the point $M_0\{r_{2p+2,2p+2}(\lambda_0), 0\}$ and the axis X_{2p+1}, i.e. the straight line $y = a_{2p+2}$ at the point $P_0\{r_{2p+1,2p+1}(\lambda_0), a_{2p+2}\}$. We shall now drawn the straight line $x = r_{2p+1,2p+1}(\lambda_0)$. This straight line cuts the axis X_{2p+2} at the point N_0, the axis X_{2p+1} at the point P_0 and curve (78) at the point K_0. Since the arc of curve (78) for $x \in (r_{2p+2,2p+2}(\lambda_0), +\infty)$ lies above the arc of curve (79), the point K_0 is situated above the axis X_{2p+1}, thus the straight line segment N_0K_0 is larger than the straight line segment $N_0P_0 = a_{2p+2}$. Through the point K_0 we shall drawn a parallel to the axis

X_{2p+2}. We call this parallel the axis $X_{2p+3}(\lambda_0)$. The ordinate of the point K_0 in the system (X_{2p+2}) is

(80)
$$N_0K_0 = {}_{2p+3}y_{2p+2}\{r_{2p+1,2p+1}(\lambda_0), \lambda_0\} = r_{2p+1,2p+1}(\lambda_0)\cdot y_{2p+2}\{r_{2p+1,2p+1}(\lambda_0)\lambda_0\} > 0.$$

By $a^0_{2p+3}(\lambda_0)$ we denote the absolute value of expression (80)

(81) $$N_0K_0 = a^0_{2+3}(\lambda_0) = {}_{2p+3}y_{2p+2}\{r_{2p+1,2p+1}(\lambda_0),\ \lambda_0\} > 0.$$

We consider the polynomial of degree $(2p+3)$

(82) $$y_{2p+3}(x,\ \lambda_0) = a_0(\lambda_0)x^{2p+3} - a_1x^{2p+2} \pm \ldots + a_{2p+2}x - a_{2p+3},$$

in which the coefficient $a_0(\lambda_0)$ has a constant value, and for the coefficient a_{2p+3} we assume that it is situated in the interval

(83) $$0 < a_{2p+3} < a^0_{2p+3}(\lambda_0).$$

By $\bar{X}_{2p+3}(a_{2p+3})$ we denote the axis of abscissas of the system (X_{2p+3}) when a_{2p+3} satisfies condition (83). This axis lies between the axes X_{2p+2} and $X_{2p+3}(\lambda_0)$. Provided that condition (83) is fulfilled, the right branch of the last arc of parabola (82) cuts the axis $\bar{X}_{2p+3}(a_{2p+3})$ at the point L_0; the point L_0 varies along the arc M_0K_0 as a_{2p+3} varies in the interval (83). Since curve (82) is increasing in the interval $(r_{2p+2,2p+2}(\lambda_0), +\infty)$, it follows, owing to (83)

$$r_{2p+2,2p+2}(\lambda_0) < r_{2p+3,2p+3}(\lambda_0,\ a_{2p+3}),$$

where by $r_{2p+3,2p+3}(\lambda_0,\ a_{2p+3})$ we have denoted that for each special value a_{2p+3} from the interval (83) we have a certain other value $r_{2p+3,2p+3}(\lambda_0,\ a_{2p+3})$. It is seen that $r_{2p+3,2p+3}(\lambda_0,\ a_{2p+3})$ varies in the interval $(r_{2p+2,2p+2}(\lambda_0), r_{2p+1,2p+1}(\lambda_0))$ as a_{2p+3} varies in the interval (83), thus we have

(84) $$0 < r_{2p+2,2p+2}(\lambda_0) < r_{2p+3,2p+3}(\lambda_0,\ a_{2p+3}) < r_{2p+1,2p+1}(\lambda_0)$$

provided $0 < a_{2p+3} < a^0_{2p+3}(\lambda_0)$.

In the following investigation we shall consider the position of the zero $r_{2p+3,2p+3}(\lambda)$ in relation to the position of the zero $r_{2p+1,2p+1}(\lambda)$, for $\lambda > \lambda_0$. If in the equation of curve (82) we take $a_{2p+3} \equiv a^0_{2p+3}(\lambda_0)$, then the curve

(85) $$y = y_{2p+3}\{x,\ a^0_{2p+3}(\lambda_0)\} \equiv x\cdot y_{2p+2}(x, \lambda_0) - a^0_{2p+3}(\lambda_0)$$

cuts the axis $X_{2p+3}(\lambda_0)$ at the point K_0 whose abscissa is $r_{2p+1,2p+1}(\lambda_0)$; hence $r_{2p+1,2p+1}(\lambda_0)$ is the greatest zero of polynomial (85); we thus have

(86) $$r_{2p+3,2p+3}\{\lambda_0,\ a^0_{2p+3}(\lambda_0)\} \equiv r_{2p+1,2p+1}(\lambda_0)$$

and therefore

(87) $$y_{2p+3}\{r_{2p+1,2p+1}(\lambda_0),\ a^0_{2p+3}(\lambda_0)\} \equiv 0.$$

From (87) it follows

$$a^0_{2p+3}(\lambda_0) = r_{2p+1,2p+1}(\lambda_0)\cdot\{r_{2p+1,2p+1}(\lambda_0)\cdot y_{2p+1}[r_{2p+1,2p+1}(\lambda_0),\ \lambda_0] + a_{2p+2}\}.$$

Since $y_{2p+1}[r_{2p+1,2p+1}(\lambda_0),\ \lambda_0] \equiv 0$, we obtain

$$a^0_{2p+3}(\lambda_0) = r_{2p+1,2p+1}(\lambda_0)\cdot a_{2p+2}.$$

Therefore the length of the straight line segment N_0K_0 is given by the formula

(88) $$N_0K_0 = a^0_{2p+3}(\lambda_0) = r_{2p+1,2p+1}(\lambda_0)\cdot a_{2p+2}.$$

For $\lambda_1 > \lambda_0$ we obtain in the same manner the point K_1, the axis $X_{2p+3}(\lambda_1)$ and the straight line segment

(89) $$N_1K_1 = a^0_{2p+3}(\lambda_1) = r_{2p+1,2p+1}(\lambda_1)\cdot a_{2p+2}.$$

From (88) and (89) it follows

(90) $$a^0_{2p+3}(\lambda_1) > a^0_{2p+3}(\lambda_0),\quad \lambda_1 > \lambda_0,$$

i.e. the axis $X_{2p+3}(\lambda_1)$ is above the axis $X_{2p+3}(\lambda_0)$. Also for $\lambda_2 > \lambda_1 > \lambda_0$ we obtain the point K_2, the axis $X_{2p+3}(\lambda_2)$ and the inequality

$$a^0_{2p+3}(\lambda_2) > a^0_{2p+3}(\lambda_1) > a^0_{2p+3}(\lambda_0).$$

For an arbitrary value $\lambda > \lambda_0$ we have the point $K(\lambda)$ and the straight line segment $NK(\lambda)$,

(91) $$NK(\lambda) = a^0_{2p+3}(\lambda) = r_{2p+1,2p+1}(\lambda)\cdot a_{2p+2}.$$ The curve

(92) $$y = r_{2p+1,2p+1}(\lambda)\cdot a_{2p+2},$$

on which the points $K(\lambda)$ are situated, is monotonously increasing, since, by theorem [1] (4) and by formula [1] (53), the last zero $r_{2p+1,2p+1}(\lambda)$ increases monotonously with λ.

Between the axes X_{2p+2} and $X_{2p+3}(\lambda_0)$ and to the right of the arc M_0K_0 there is the domain D_0 of the plane XY. Between the axes X_{2p+2} and $X_{2p+3}(\lambda_1)$ and to the right of the arc M_1K_1 there is the domain D_1 and so on. In the same manner to each real number $\lambda > \lambda_0$ there belongs the domain $D(\lambda)$.

Let us now consider the family of parabolas of degree $(2p+3)$

(93) $$y = y_{2p+3}(x,\ \lambda) = a_0(\lambda)x^{2p+3} - a_1x^{2p+2} \pm \ldots + a_{2p+2}x - a_{2p+3}$$

in which the coefficient a_{2p+3} is constant. Let us denote by $\bar{X}_{2p+3}(a_{2p+3})$ the axis of abscissas belonging to the constant a_{2p+3}, if $0 < a_{2p+3} < a^0_{2p+3}(\lambda_0)$ and by $\bar{\bar{X}}_{2p+3}(a_{2p+3})$ if $a_{2p+3} > a^0_{2p+3}(\lambda_0)$.

1° Let be $0 < a_{2p+3} < a^0_{2p+3}(\lambda_0)$; then, for $\lambda = \lambda_0$, there holds (84). If $\lambda_1 > \lambda_0$, the right branch of the last arc of the parabola $y = y_{2p+3}(x,\ \lambda_1)$ passes through the point $K(\lambda_1)$ lying above the axis $X_{2p+3}(\lambda_0)$ and cuts the axis $\bar{X}_{2p+3}(a_{2p+3})$ being in the domain $D(\lambda_0)$, at the point $r_{2p+3,2p+3}(\lambda_1)$ for which there is

(94) $$0 < r_{2p+2,2p+2}(\lambda_1) < r_{2p+3,2p+3}(\lambda_1) < r_{2p+1,2p+1}(\lambda_1)$$

provided $\lambda_1 > \lambda_0,\ 0 < a_{2p+3} \leqslant a^0_{2p+3}(\lambda_0)$.
Consequently inequality (94) is valid in the domain $D(\lambda_0)$ for each $\lambda >$ $> \lambda_0$.

2° Let be $a_{2p+3} > a^0_{2p+3}(\lambda_0)$; then by (90), there exists a certain $\lambda' > \lambda_0$ for which, by (91), there is

(95) $$a_{2p+3} \equiv a^0_{2p+3}(\lambda') = r_{2p+1,2p+1}(\lambda')\cdot a_{2p+2},\quad \lambda' > \lambda_0.$$

The right branch of the last arc of the paraboly $y=y_{2p+3}(x, \lambda')$ passe through the point $K(\lambda')$ lying on the axis $X_{2p+3}(\lambda')\equiv \bar{\bar{X}}_{2p+3}(a_{2p+3} > a^0_{2p+3}(\lambda_0))$. For $\lambda_2 > \lambda' > \lambda_0$ the point $K(\lambda_2)$ is situated above the axis $X_{2p+3}(\lambda')$; hence

$$(96) \qquad 0 < r_{2p+2,2p+2}(\lambda_2) < r_{2p+3,2p+3}(\lambda_2) < r_{2p+1,2p+1}(\lambda_2)$$

provided $\lambda_2 > \lambda' > \lambda_0$, $a_{2p+3} \equiv a^0_{2p+3}(\lambda') > a^0_{2p+3}(\lambda_0)$.
Inequality (96) is valid in relation to each axis of abscissas being below the axis $X_{2p+3}(\lambda')$, i.e. inequality (96) is valid also for $0 < a_{2p+3} \leqslant a^0_{2p+3}(\lambda_0)$. Inequality (96) is valid for each $\lambda < \lambda'$ and for $0 < a < a_{2p+3} \leqslant a^0_{2p+3}(\lambda')$, i.e. it is valid in the domain $D(\lambda')$. Thus, for an arbitrary a_{2p+3}, there exists such a value λ' that for $\lambda > \lambda'$ the inequality

$$(97) \qquad 0 < r_{2p+2,2p+2}(\lambda) < r_{2p+3,2p+3}(\lambda) < r_{2p+1,2p+1}(\lambda)$$

is satisfied for $\lambda > \lambda' > \lambda_0$, $0 < a_{2p+3} \leqslant a^0_{2p+3}(\lambda')$.

The magnitude of the number α_{2p+2} depends on the magnitude of the coefficient a_{2p+2}. If $a_{2p+2} > |_{2p+2}M_{2p+1}(\lambda^{(2p)})|$, then, by (73), the right branch of the last arc of the curve $y=y_{2p+2}(x, \lambda)$ cuts the axis X_{2p+2} if $\lambda > \bar{\lambda} > \lambda_0 \geqslant \alpha_{2p+2} > \lambda^{(2p)}$, and, by (97), the same arc cuts the axis X_{2p+2} if $\lambda > \lambda' > \lambda_0 \geqslant \alpha_{2p+2} > \lambda^{(2p)}$. In virtue of these conditions which determine the lower limit for λ, between $\bar{\lambda}$ and λ' the greater value should be chosen. If we denote this value by λ_{2p+3}, then inequalities (73) and (97) give

$$(98) \qquad \begin{aligned} &0 < r_{2p,2p}(\lambda) < r_{2p+2,2p+2}(\lambda) < r_{2p+3,2p+3}(\lambda) < r_{2p+1,2p+1}(\lambda), \\ &\lambda > \lambda_{2p+3} > \lambda_0 \geqslant \alpha_{2p+2} > \lambda^{(2p)}, \quad a_{2p+2} > |_{2p+2}M_{2p+1}(\lambda^{(2p)})|. \end{aligned}$$

If in (98) in place of p we insert the numbers $1, 2, 3, \ldots, p-1$, we obtain the distribution of the zeros (2) mentioned in the introduction. Let us remark that $r_{22}(\lambda)$ is the zero of greater absolute value of the trinomial $a_0(\lambda)x^2 - a_1x + a_2$.

From [1] (83) and (2) it follows that the roots tending to $+\infty$ of equations (1) lie in the interval (r_{22}, r_{33}) which tends to zero with a_0.

In the interval (r_{22}, r_{33}) at a finite distance from the origin on the positive part of the X-axis there is only a finite number of zeros of the sequence (2), since increasing the number of terms in the sequence (2) requires diminishing the parameter a_0. However, this does not hold also conversely: when diminishing the parameter a_0 it is not necessary to increase the number of terms in the sequence (2).

For the sequence of real equations

$$(99) \qquad \sum_{\nu=0}^{p} a_\nu x^{p-\nu} = 0, \quad p = 2,3,4,\ldots,n, \quad (a_0 \to 0)$$

with positive coefficients it can be proved by means of the same method [2] that the negative roots of the greatest absolute value tend to $-\infty$ as $a_0 \to 0$, and that they are distributed as follows

$$(100) \quad r_{31}(\lambda) < r_{51}(\lambda) < \ldots < r_{2p+1,1}(\lambda) < r_{2p,1} < \ldots < r_{41}(\lambda) < r_{21}(\lambda) < 0.$$

Let us remark here also that $r_{21}(\lambda)$ is the zero of greater absolute value of the trinomial $a_0(\lambda)x^2 + a_1x + a_2$. For the trinomials $a_0(\lambda)x^2 + a_1x - a_2$ and $a_0(\lambda)x^2 - a_1x - a_2$ the same remark is valid.

The final conclusion is: For equations (1) and (99), provided $a_0 > 0$ it is the sign of $a_1 > 0$ which determines whether the root of the greatest absolute value will tend to $+\infty$ or to $-\infty$ when $a_0 \to 0$.

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