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ON A METHOD FOR APPROXIMATE SOLVING OF NON-LINEAR OPERATIONAL EQUATIONS

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The approximate methods for solving non-linear operational equations following the Newton-like iterative process, the method of chords and all its variants, assume the existence and the continuity of the inverse of a linear operator (derivative, divided difference). But the examination of the existence of this inverse operator and the evaluation of its norm are the greatest difficulties.

In the present paper we give an iterative method for approximate solutions of non-linear operational equations in Hilbert real spaces, which eliminates the above-mentioned difficulties.

Let H be a Hilbert real space and let $P : H \to H$ be a continuous operator. Consider the non-linear operational equation

(1) $$P(x) = 0$$

and the equivalent equation

(2) $$\|P(x)\|^2 = \langle P(x), P(x)\rangle = 0.$$

The *divided difference* $[u, v; \|P\|^2]$ of the functional $\|P\|^2 : H \to R$ for the points $x', x'' \in H$, $x' \neq x''$ is a linear mapping given by [2]:

(3) $$[u, v; \|P\|^2] = \langle [u, v; P], P(u) + P(v)\rangle$$

where $[u, v; P]$ is a divided difference for the operator P. We observe that every operator $P : H \to H$ has divided difference and that there exists at least one with the following property [4]:

(4) $$\|[u, v; P]\| = \frac{\|P(u) - P(v)\|}{\|u - v\|}.$$

In the case of the functions $f : D \to R$ ($D \subseteq R^n$), the divided difference, defined by

$$[f(u) - f(v)]\left(\frac{u_1 - v_1}{\|u - v\|^2}, \ldots, \frac{u_n - v_n}{\|u - v\|^2}\right),$$

where $u = (u_1, u_2, \ldots, u_n)^*$ and $v = (v_1, v_2, \ldots, v_n)^*$, has the property (4). In the case of the functions $f: D \to R^m (D \infty R^n)$ such a divided difference is given by matrix

$$\left([f_j(u) - f_j(v)] \frac{u_i - v_i}{\|u - v\|^2}\right)_{m,n}, \quad i = \overline{1,n};\ j = \overline{1,m}.$$

If (x_n) is a sequence of the space H, we choose a sequence of elements $z_n \in H$ such that

(5) $\qquad \|z_n\| = 1$ and $|[x_n, x_{n-1}; \|P\|^2](z_n)| = \|[x_n, x_{n-1}; \|P\|^2]\|$.

Let us note that such a choice is possible if we consider the divided difference which satisfies conditions (4). Indeed, if we consider

(6) $$z_n = \frac{x_n - x_{n-1}}{\|x_n - x_{n-1}\|}$$

then obviously we have $\|z_n\| = 1$ and

$$[x_n, x_{n-1}; \|P\|^2](z_n) = \langle [x_n, x_{n-1}; P](z_n), P(x_n) + P(x_{n-1})\rangle =$$
$$= \frac{1}{\|x_n - x_{n-1}\|} \langle P(x_n) - P(x_{n-1}), P(x_n) + P(x_{n-1})\rangle =$$
$$= \frac{1}{\|x_n - x_{n-1}\|} \langle P(x_n), P(x_n)\rangle - \langle P(x_{n-1}), P(x_{n-1})\rangle =$$
$$= \frac{\|P(x_n)\|^2 - \|P(x_{n-1})\|^2}{\|x_n - x_{n-1}\|}$$

from which we obtain

$$|[x_n, x_{n-1}; \|P\|^2](z_n)| = \frac{|\|P(x_n)\|^2 - \|P(x_{n-1})\|^2|}{\|x_n - x_{n-1}\|} = \|[x_n, x_{n-1}; \|P\|^2]\|,$$

therefore the statement is proved.

The divided difference of the function $\|P\|^2$ has the Liepschitz-property in $A \subseteq H$ (with respect to the second argument) if there exists $K > 0$ such that for every $u, v, w \in A$

$$\|[u, v; \|P\|^2] - [u, w; \|P\|^2]\| \leqslant K\|v - w\|,$$

that is

$$\|\langle [u, v; P], P(u) + P(v)\rangle - \langle [u, w; P], P(u) + P(w)\rangle\| \leqslant K\|v - w\|.$$

THEOREM. *Suppose that the following conditions hold:*

1° *there exist the points* $x_0, x_1 \in H$ *and the constant* $B_0 > 0$ *such that*

$$B_0\|\langle [x_0, x_{-1}; P], P(x_0) + P(x_{-1})\rangle\| \geqslant 1;$$

2° *there exist the constant* $\eta_0, \eta_0' (\eta_0 \leqslant \eta_0')$ *such that the following inequalities are satisfied:*

$$\|P(x_0)\|^2 \leqslant \eta_0\|\langle [x_0, x_{-1}; P], P(x_0) + P(x_{-1})\rangle\|, \quad \|x_0 - x_{-1}\| \leqslant \eta_0';$$

3° *the divided difference of the functional* $\|P\|^2$ *has the Liepschitz-property with the constant* K *in the ball* $S[x_0, r]$, *where* $r = \max\{\eta_0', \frac{3}{2}\eta_0\}$;

4° *the constants* B_0, η_0, η_0', K *satisfy the inequality*

$$h_0 := B_0 K(\eta_0 + \eta_0') < \frac{1}{4}.$$

Under these conditions the equality

(7) $$x_{n+1} = x_n - \frac{\|P(x_n)\|^2}{\langle [x_n, x_{n-1}; P](z_n), P(x_n) + P(x_{n-1})\rangle} z_n,$$

$n \geqslant 0$ *defines by recurence a sequence* (x_n) *having the following properties:*

a) *the sequence* (x_n) *is convergent and*

$$\lim_{n\to\infty} x_n = x^* \in S[x_0, r];$$

b) x^* *is a solution of equation* (2);

c) *for the error estimate we have the inequality*

$$\|x^* - x_n\| \leqslant \frac{1}{2^{s_n - 1}}\left(\frac{8}{9}\right)^{s_n - 1} (4h_0)^{s_n} \eta_0, \textit{ where}$$

$$s_n = \sum_{i=1}^{n} u_i, \ u_1, u_2 = 1 \textit{ and } u_{n+1} = u_n + u_{n-1},\ n \geqslant 2.$$

Remark. If we use the divided difference having property (4) and the sequence given by (6), then the equality which defines the sequence (x_n) is the following

$$x_{n+1} = x_n - \frac{\|P(x_n)\|^2}{\|P(x_n)\|^2 - \|P(x_{n-1})\|^2}(x_n - x_{n-1})$$

or

$$x_{n+1} = x_{n-1} - \frac{\|P(x_{n-1})\|^2}{\|P(x_n)\|^2 - \|P(x_{n-1})\|^2}(x_n - x_{n-1}).$$

Proof. From equality (7) we get

(8) $$[x_n, x_{n-1}; \|P\|^2](x_n - x_{n-1}) = \|P(x_n)\|^2$$

whence we have the following identity

(9) $$[x_n, x_{n-1}; \|P\|^2](x_n - x_{n-1}) = \|P(x_n)\|^2 - P(x_{n-1})\|^2 - -[x_{n-1}, x_{n-2}; \|P\|^2](x_n - x_{n-1}).$$

From the identities (8) and (9), using equality (7), we obtain

$$\|x_{n+1} - x_n\| \leqslant \frac{1}{\|[x_n, x_{n-1}; \|P\|^2]\|} |\|P(x_n)\|^2 - \|P(x_{n-1})\|^2 - -[x_{n-1}, x_{n-2}; \|P\|^2](x_n - x_{n-1})|$$

whence, using the definition of the divided difference and condition 3°, we have

$$(10)\qquad \|x_{n+1}-x_n\| \leqslant \frac{K}{\|[x_n,\ x_{n-1};\ \|P\|^2]\|}\,\|x_n - x_{n-1}\|\cdot\|x_n - x_{n-2}\|.$$

From the obvious inequality

$$\|[x_1,\ x_0;\|P\|^2]\| \geqslant \|[x_0,\ x_{-1};\ \|P\|^2]\| \times$$

$$\times\left[1-\frac{\|[x_1,\ x_0;\|P\|^2]-[x_0,\ x_{-1};\|P\|^2]\|}{\|[x_0,\ x_{-1};\|P\|^2]\|}\right],$$

using the conditions of the Theorem, we obtain (x_{-1} and x_0 are in the ball $S[x_0,\ r]$)

$$(11)\qquad \frac{1}{\|[x_1,\ x_0;\ \|P\|^2]\|} \leqslant \frac{B_0}{1-h_0} = B_1,$$

or

$$B_1\cdot\|\langle[x_1,\ x_0;\ P],\ P(x_1)+P(x_0)\rangle\| \geqslant 1.$$

In the $n=1$ case from inequality (10) using (11), we derive

$$\|x_2-x_1\| \leqslant \frac{B_0K}{1-h_0}\,\eta_0(\eta_0+\eta_0') = \frac{h_0\eta_0}{1-h_0} = \eta_1 < \frac{\eta_0}{3}$$

or $\|P(x_1)\|^2 \leqslant \eta_1\|\langle[x_1,\ x_0;P],\ P(x_1)+P(x_0)\rangle\|$.

We have

$$h_1 = B_1K(\eta_1+\eta_0) \leqslant \frac{B_0K}{1-h_0}\left(\frac{h_0\eta_0}{1-h_0}+\eta_0\right) =$$

$$= \frac{2B_0K\eta_0}{2(1-h_0)^2} \leqslant \frac{h_0}{2(1-h_0)^2} < \frac{8}{9}\,h_0 < \frac{1}{4}.$$

Therefore conditions 1°, 2° and 4° of the Theorem are satisfied for the points x_1, x_0.

By mathematical induction we shall prove the existence of the constants $B_n > 0$, $\eta_n > 0$ such that the following inequalities hold

$$(12)\qquad \|[x_n,\ x_{n-1};\ \|P\|^2]^{-1}\| \leqslant B_n = \frac{B_{n-1}}{1-h_{n-1}},$$

$$(13)\qquad \|x_{n+1}-x_n\| \leqslant \frac{h_{n-1}\eta_{n-1}}{1-h_{n-1}} = \eta_n < \frac{\eta_0}{3^n},$$

$$(14)\qquad h_n = B_nK(\eta_n+\eta_{n-1}) \leqslant \frac{h_{n-1}h_{n-2}}{(1-h_{n-1})^2} < \frac{1}{4}.$$

From inequalities (13) and (14), we obtain

$$h_n \leqslant \frac{1}{2}\left(\frac{8}{9}\right)^{s_n}(4h_0)^{s_{n+1}}$$

$$(15)\qquad \eta_n \leqslant \left(\frac{2}{3}\right)^n\left(\frac{8}{9}\right)^{s_{n-1}}(4h_0)^{s_n}\cdot\eta_0$$

Using inequality (15), we have

$$\|x_{n+p}-x_n\| \leqslant \frac{1}{2^{s_{n-1}}}\left(\frac{8}{9}\right)^{s_{n-1}}(4h_0)^{s_n}\cdot\eta_0$$

for every $n,\ p \in N$. Because the space H is a Banach space, it results the existence of the limit of the sequence (x_n), and $\lim\limits_{n\to\infty} x_n = x_* \in S[x_0,\ r]$.

From the previous inequality for p we obtain the formula which we have for the error estimate.

Using inequality (13), we get that the elements of the sequence (x_n) and x^* are in the ball $S[x_0,\ r]$. Indeed, we have

$$\|x_n - x_0\| \leqslant \|x_0 - x_1\| + \|x_1 - x_2\| + \ldots + \|x_{n-1}-x_n\| \leqslant$$

$$\leqslant \eta_0 + \frac{\eta_0}{3} + \ldots + \frac{\eta_0}{3^{n-1}} < \frac{3}{2}\,\eta_0.$$

From (8) it follows that x^* is a solution of equation (2) if the linear functionals $[x_n,\ x_{n+1};\ \|P\|^2]$ are bounded.

The boundedness of $[x_n,\ x_{n-1};\|P\|^2]$ results from the following evident equality:

$$[x_n,\ x_{n-1};\|P\|^2] = [x_n,\ x_{n-1};\ \ \|P\|^2] - [x_{n-1},\ x_{n-2};\ \ \|P\|^2] +$$

$$+[x_{n-1},\ x_{n-2};\|P\|^2] - [x_{n-2},\ x_{n-3};\|P\|^2] + \ldots + [x_1,\ x_0;\ \|P\|^2]$$

We have

$$\|[x_n,\ x_{n-1};\ \ \|P\|^2]\| \leqslant K(\|x_n - x_{n-2}\| + \|x_{n-1}-x_{n-3}\| + \ldots + \|x_2 - x_0\|)$$

$$+\|[x_0,\ x_1;\ \ \|P\|^2]\| \leqslant 3K\eta_0 + \|[x_0,\ x_1;\ \ \|P\|^2]\| \leqslant M$$

Remark. Particularly, for $P: I\to R$ ($I \subseteq R$ is an interval), if the points $x_0,\ x_1 \in I$ exist and the positive constants: $B_0,\ \eta_0,\ \eta_0',\ K$ are such that the following conditions are satisfied:

1° $|x_0 - x_1| \leqslant B_0|P^2(x_0) - P^2(x_{-1})|$;

2° $P^2(x_0)|x_0 - x_{-1}| \leqslant \eta_0|P^2(x_0) - P^2(x_{-1})|, \qquad |x_0 - x_1| \leqslant \eta_0'$;

$$3°\ \left|\frac{P(u)-P(v)}{u-v}\,[P(u)+P(v)] - \frac{P(u)-P(w)}{u-w}\,[P(u)+(P(w)]\right| \leqslant$$

$$\leqslant K|u-w|$$

for every $u, v, w \in I_0 = [x_0 - r,\ x_0 + r]$, where $r = \max\left\{\eta_0', \dfrac{3}{2}\eta_0\right\}$;

4° $h_0 = B_0 K(\eta_0 + \eta_0') < \dfrac{1}{4}$,

then the sequence (x_n) given by the equality

$$x_{n+1} = x_n - \frac{P^2(x_n)}{P^2(x_n) - P^2(x_{n-1})}(x_n - x_{n-1}) \quad (n \geq 0)$$

converges, $\lim\limits_{n\to\infty} x_n = x^* \in I_0$, x^* is a solution of the equation $P(x) = 0$ and we have

$$|x^* - x_n| \leq \frac{1}{2^{s_n - 1}}\left(\frac{8}{9}\right)^{s_{n-1}} \cdot (4h_0)^{s_n}\eta_0.$$

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