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SPLINE APPROXIMATIONS FOR SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS, III

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The purpose of this paper is to present a method for approximating the solution of the system of non-linear ordinary differential equations $y' = f_1(x, y, z)$, $z' = f_2(x, y, z)$ with $y(x_0) = y_0$ and $z(x_0) = z_0$ We use spline functions, which are not necessarily polynomial splines, for finding the approximate solution. It is a one-step method of $O(h^{\alpha+r+1})$ in $y^{(q)}(x)$ and $z^{(q)}(x)$, $q = 0(1)r+1$ where $0 < \alpha \leqslant 1$, assuming that $f_1, f_2 \in C^r[a, b]$, $r \in I^+$.

Description of the method. Consider the system of ordinary differential equations

(1) $$y' = f_1(x, y, z), \quad y(x_0) = y_0,$$

(2) $$z' = f_2(x, y, z), \quad z(x_0) = z_0,$$

where $f_1, f_2 \in C^r([0, 1] \times R^2)$.

Let Δ be the partition

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1$$

where $x_{k+1} - x_k = h < 1$ and $k = 0(1)n - 1$.

Let L_1 and L_2 be the Lipschitz constants satisfied by the functions $f_1^{(q)}$ and $f_2^{(q)}$ respectively, i.e.,

(3) $$|f_1^{(q)}(x, y_1, z_1) - f_1^{(q)}(x, y_2, z_2)| \leqslant L_1\{|y_1 - y_2| + |z_1 - z_2|\},$$

(4) $$|f_2^{(q)}(x, y_1, z_1) - f_2^{(q)}(x, y_2, z_2)| \leqslant L_2\{|L_2|y_1 - y_2| + |z_1 - z_2|\}$$

for all (x, y_1, z_1) and (x, y_2, z_2) in the domain of definition of f_1 and f_2 and all $q = 0(1)r$.

It should be noted that we use the Lipschitz conditions on f_1 and f_2 to guarantee the existence of a unique solution to problem (1)—(2).

The functions $f_1^{(q)}$ and $f_2^{(q)}$, $q = 0(1)r$ are functions of x, y and z only and they are given from the following Algorithm :

Let $f_1^{(0)} = f_1(x, y, z)$,
and $f_2^{(0)} = f_2(x, y, z)$.

Then, for all $q = 0(1)r$,

$$y^{(q+1)} = \frac{d^q f_1}{dx^q} = f_1^{(q)} = \frac{\partial f_1^{(q-1)}}{\partial x} + \frac{\partial f_1^{(q-1)}}{\partial y} f_1 + \frac{\partial f_1^{(q-q)}}{\partial z} f_2$$

and

$$z^{(q+1)} = \frac{d^q f_2}{dx^q} = f_2^{(q)} = \frac{\partial f_2^{(q-1)}}{\partial x} + \frac{\partial f_2^{(q-1)}}{\partial y} f_1 + \frac{\partial f_2^{(q-1)}}{\partial z} f_2.$$

Then, we define the spline functions approximating the solution $y(x)$ and $z(x)$ by $S_\Delta(x)$ and $\bar{S}_\Delta(x)$, where

$$(5) \qquad S_\Delta(x) = S_k(x), \quad x_k \leqslant x \leqslant x_{k+1}, \quad k = 0(1)n-1$$

and

$$(6) \qquad \bar{S}_\Delta(x) = \bar{S}_k(x), \quad x_k \leqslant x \leqslant x_{k+1}, \quad k = 0(1)n-1.$$

Both $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ are given from the following :

$$S_k(x) = S_{k-1}(x_k) + \int_{x_k}^{x} f_1[t,\ S_{k-1}(x_k) + \sum_{j=0}^{r} \frac{(t-x_k)^{j+1}}{(j+1)!} \cdot$$

$$(7) \qquad f_1^{(j)}\{x_k,\ S_{k-1}(x_k),\ \bar{S}_{k-1}(x_k)\},\ \bar{S}_{k-1}(x_k) +$$

$$+ \sum_{j=0}^{r} \frac{(t-x_k)^{j+1}}{(j+1)!} f_2^{(j)}\{x_k,\ S_{k-1}(x_k),\ \bar{S}_{k-1}(x_k)\}]\, dt \ldots$$

and

$$\bar{S}_k(x) = \bar{S}_{k-1}(x_k) + \int_{x_k}^{x} f_2\Big[t,\ S_{k-1}(x_k) + \sum_{j=0}^{r} \frac{(t-x_k)^{j+1}}{(j+1)!} \cdot$$

$$(8) \qquad f_1^{(j)}\{x_k,\ S_{k-1}(x_k),\ \bar{S}_{k-1}(x_k)\},\ \bar{S}_{k-1}(x_k) +$$

$$+ \sum_{j=0}^{r} \frac{(t-x_k)^{j+1}}{(j+1)!} f_2^{(j)}\{x_k,\ \bar{S}_{k-1}(x_k)\},\ \bar{S}_{k-1}(x_k)\}\Big]\, dt \ldots$$

where $k = 0(1)n-1$ and $S_{-1}(x_0) = y_0$, $\bar{S}_{-1}(x_0) = z_0$.

By construction, it is clear that $S_\Delta(x)$ and $S_\Delta(x) \in C[0, 1]$.

Convergence. We are going to discuss the convergence of these spline approximants.

For all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$ the exact solutions of (1) and (2) can be written — by means of Taylor's expansions — in the forms :

$$(9) \qquad y(x) = y_k + \int_{x_k}^{x} f_1\left[t, \sum_{j=0}^{r+1} \frac{(t-x_k)^j}{j!} y_k^{(j)}, \sum_{j=0}^{r+1} \frac{(t-x_k)^j}{j!} z_k^{(j)}\right] dt$$

and

$$(10) \qquad z(x) = z_k + \int_{x_k}^{x} f_2\left[t, \sum_{j=0}^{r+1} \frac{(t-x_k)^j}{j!} y_k^{(j)}, \sum_{j=0}^{r+1} \frac{(t-x_k)^j}{j!} z_k^{(j)}\right] dt,$$

where $y_k^{(r+1)} = y^{(r+1)}(\xi_k)$, $z_k^{(r+1)} = z^{(r+1)}(\eta_k)$, ξ_k, $\eta_k \in (x_k, x_{k+1})$ and $k = 0(1)n-1$.

First, we estimate $|y(x) - S_0(x)|$, where $x_0 \leqslant x \leqslant x_1$.

Using both (7) and (9) for $k = 0$ and the Lipschitz condition (3), we get :

$$|y(x) - S_0(x)| \leqslant L_1 \int_{x_0}^{x} \Big[\{|y^{(r+1)}(\xi_0) - y_0^{(r+1)}| + |z^{(r+1)}(\eta_0) - z_0^{(r+1)}|\} \cdot \frac{|t-x_0|^{r+1}}{(r+1)!}\Big]\, dt$$

$$(11) \qquad \leqslant L_1 \frac{h^{r+2}}{(r+2)!}\{\omega(y^{(r+1)}, h) + \omega(z^{(r+1)}, h)\}$$

$$\leqslant \frac{2L_1}{(r+2)!} h^{r+2}\omega(h) = 0(h^{\alpha+r+2})$$

where $\omega(y^{(r+1)}, h)$ and $\omega(z^{(r+1)}, h)$ are the moduli of continuity of the functions $y^{(r+1)}$ and $z^{(r+1)}$ respectively, and

$$(12) \qquad \omega(h) = \max\ \{\omega(y^{(r+1)}, h),\ \omega(z^{(r+1)}, h)\}.$$

Also, we estimate $|y'(x) - S_0'(x)|$.

Using both (7) and (9) for $k = 0$ and making use of (3), we get :

$$(13) \qquad |y'(x) - S_0'(x)| \leqslant L_1\{|y^{(r+1)}(\xi_0) - y_0^{(r+1)}| + |z^{(r+1)}(\eta_0) - z_0^{(r+1)}|\}.$$

$$\cdot \frac{|x-x_0|^{r+1}}{(r+1)!} \leqslant \frac{2L_1}{(r+1)!} h^{r+1}\omega(h) = 0(h^{\alpha+r+1})$$

To estimate $|y''(x) - S_0''(x)|$ we use both (7) and (9) setting $k = 0$ and the Lipschitz condition (3), so that we get :

$$(14) \qquad |y''(x) - S_0''(x)| \leqslant \frac{2L_1}{(r+1)!} h^{r+1}\omega(h) = 0(h^{\alpha+r+1})$$

Moreover, it can be easily shown, for $q = 3(1)r+1$, that :

$$(15) \qquad |y^{(q)}(x) - S_0^{(q)}(x)| \leqslant \frac{2L_1}{(r+1)!} h^{r+1}\omega(h) = 0(h^{\alpha+r+1})$$

In a similar manner, using both (8) and (10) for $k = 0$ and the Lipschitz condition (4), it can be shown, for $x_0 \leqslant x \leqslant x_1$, that :

$$(16) \qquad |z(x) - \bar{S}_0(x)| \leqslant \frac{2L_2}{(r+2)!} h^{r+2}\omega(h) = 0(h^{\alpha+r+2}),$$

and

$$(17) \qquad |z^{(i)}(x) - \bar{S}_k^{(i)}(x)| \leqslant \frac{2L_2}{(r+1)!} h^{r+1}\omega(h) = 0(h^{\alpha+r+1}),$$

where $i = 1(1)r+1$.

In what follows we deal with the general subinterval $I_k = [x_k, x_{k+1}]$, $k = 1(1)n - 1$.

Using (7), (9) and the Lipschitz condition (3), we get :

$$(18) \qquad |y(x) - S_k(x)| \leqslant |y_k - S_{k-1}(x_k)| + L_1 \int_{x_k}^{x} [|y_k - S_{k-1}(x_k)| +$$

$$+ \sum_{j=0}^{r-1} \frac{|t - x_k|^{j+1}}{(j+1)!} |y_k^{(j+1)} - f_1^{(j)}\{x_k, S_{k-1}(x_k),$$

$$, \bar{S}_{k-1}(x_k)\}| + \frac{|t - x_k|^{r+1}}{(r+1)!} |y^{(r+1)}(\xi_k) - f_1^{(r)}\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}| +$$

$$+ |z_k - \bar{S}_{k-1}(x_k)| + \sum_{j=0}^{r-1} \frac{|t - x_k|^{j+1}}{(j+1)!} |z_k^{(j+1)} - f_2^{(j)}\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}| +$$

$$+ \frac{|t - x_k|^{r+1}}{(r+1)!} \cdot |z^{r+1}(\eta_k) - f_2^{(r)}\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}|] \, dt$$

Not, let

$$U_1 = |y_k^{(j+1)} - f_1^{(j)}\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}|.$$

Then, using the Lipschitz condition (3), we get :

$$(19) \qquad U_1 = |f_1^{(j)}(x_k, y_k, z_k) - f_1^{(j)}\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}|$$
$$\leqslant L_1\{|y_k - S_{k-1}(x_k)| + |z_k - \bar{S}_{k-1}(x_k)|\}.$$

Using the fact that $S_\Delta(x) \in C[0, 1]$, $\bar{S}_\Delta(x) \in C[0, 1]$ and the notations :

$$(20) \qquad e(x) = |y(x) - S_k(x)|$$
$$e_k(x) = |y_k - S_k(x_k)|$$
$$\bar{e}(x) = |z(x) - \bar{S}_k(x)|,$$
$$\bar{e}_k(x) = |z_k - \bar{S}_k(x_k)|$$

then, (19) becomes

$$(21) \qquad U_1 \leqslant L_1(e_k + \bar{e}_k).$$

Similarly, if we let

$$V_1 = |y^{(r+1)}(\xi_k) - f_1^{(r)}\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}|.$$

Then,

$$(22) \qquad V_1 \leqslant |y^{(r+1)}(\xi_k) - y_k^{r+1}| + |f_1^{(r)}(x_k, y_k, z_k) -$$
$$- f_1^{(r)}\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}| \leqslant \omega(y^{(r+1)}, h) + L_1(e_k + \bar{e}_k)$$

Thus, using the same procedure, it can be easily shown that :

$$(23) \qquad U_2 \equiv |z_k^{(j+1)} - f_2^{(j)}\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}| \leqslant L_2(e_k + \bar{e}_k),$$

and

$$(24) \qquad V_2 \equiv |z^{(r+1)}(\eta_k) - f_2^{(r)}\{x_k, S_{k-1}(x_k), \bar{S}_{k-1}(x_k)\}| \leqslant \omega(z^{(r+1)}, h) +$$
$$+ L_2(e_k + \bar{e}_k)$$

Using (18—24), we easily get :

$$(25) \qquad e(x) \equiv |y(x) - S_k(x)| \leqslant e_k + L_1 \Bigg[he_k + L_1(e_k + \bar{e}_k) \sum_{j=0}^{r-1} \frac{h^{j+2}}{(j+2)!} +$$
$$+ \frac{h^{r+2}}{(r+2)!}\{\omega(y^{(r+1)}, h) + L_1(e_k + \bar{e}_k)\} + h\bar{e}_k +$$
$$+ L_2(e_k + \bar{e}_k) \sum_{j=0}^{r-1} \frac{h^{j+2}}{(j+2)!} + \frac{h^{r+2}}{(r+2)!}\{\omega(z^{(r+1)}, h) + L_2(e_k + \bar{e}_k)\} \Bigg]$$

Noting that

$$\sum_{j=0}^{r-1} \frac{h^{j+1}}{(j+2)!} \leqslant e^h - 1 < e - 1$$

then, we can easily get :

$$(26) \qquad e(x) \leqslant (1 + c_0 h)e_k + c_0 h\bar{e}_k + 2L_1 \frac{h^{r+2}}{(r+2)!} \omega(h)$$

where $c_0 = L_1 + L_1^2(e - 1) + \frac{L_1^2}{(r+2)!} + L_1 L_2(e - 1) + \frac{L_1 L_2}{(r+2)!}$, is a constant independent of h.

Similarly, using (8), (10) and the Lipschitz condition (4), we can see that :

$$(27) \qquad \bar{e}(x) \leqslant c_1 h e_k + (1 + c_1 h)\bar{e}_k + 2L_2 \frac{h^{r+2}}{(r+2)!} \omega(h)$$

where $c_1 = L_2 + L_2^2(e - 1) + \frac{L_2^2}{(r+2)!} + L_1 L_2(e - 1) + \frac{L_1 L_2}{(r+2)!}$, is a constant independent of h.

To complete the proof of the convergence, we use the matrix inequality which is given in the following definition.

Definition. Let $A = [a_{ij}]$, $B = [b_{ij}]$ be two matrices of the same order; then we say that $A \leqslant B$ iff

(i) both a_{ij} and b_{ij} are non-negative.

(ii) $a_{ij} \leqslant b_{ij}$ $\forall i, j$.

In view of this definition, and if we use the matrix notations

$$E(x) = (e(x) \;\; \bar{e}(x))^T$$

$$E_k = (e_k \;\; \bar{e}_k)^T, \quad k = 0(1)n - 1$$

we can write the estimations (26) and (27) in the form :

$$E_{k+1} \leqslant \begin{pmatrix} 1 + c_0 h & c_0 h \\ c_1 h & 1 + c_1 h \end{pmatrix} \begin{pmatrix} e_k \\ \bar{e}_k \end{pmatrix} + 2 \frac{h^{r+2}}{(r+2)!} \omega(h) \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}.$$

Thus,

$$E_{k+1} \leqslant (I + hA) E_k + \frac{2h^{r+2}}{(r+2)!} \omega(h)\ B, \tag{28}$$

where $A = \begin{pmatrix} c_0 & c_0 \\ c_1 & c_1 \end{pmatrix}$, $B = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$ and I is the identity matrix of order 2.

Now, we give the following definition of the matrix norm :

Let $T = [t_{ij}]$ be an $m \times n$ matrix ; then we define :

$$\|T\| = \max_{i,j} |t_{ij}|.$$

Using this definition, we get :

$$\|E_k\| = \max\ (e_k, \bar{e}_k), \quad k = 0(1)n - 1. \tag{29}$$

Since (28) is valid for all $x[x_k, x_{k+1}]$, $k = 0(1)\ n-1$, then the following inequalities hold true :

$$\|E(x)\| \leqslant (1 + h\|A\|)\ \|E_k\| + \frac{2h^{r+2}}{(r+2)!} \omega(h)\ \|B\|$$

$$(1 + h\|A\|)\ \|E_k\| \leqslant (1 + h\|A\|)^2\ \|E_{k-1}\| + \frac{2h^{r+2}}{(r+2)!} \omega(h)\ (1 + h\|A\|)\|B\|$$

$$(1 + h\|A\|)^2 \|E_{k-1}\| \leqslant (1 + h\|A\|)^3 \|E_{k-2}\| + \frac{2h^{r+2}}{(r+2)!} \omega(h)(1 + h\|A\|)^2 \|B\|$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$(1 + h\|A\|)^k\ \|E_1\| \leqslant (1 + h\|A\|)^{k+1} \|E_0\| + \frac{2h^{r+2}}{(r+2)!} \omega(h)\ (1 + h\|A\|)^k\ \|B\|$$

Adding L.H.S. and R.H.S. in these inequalities, and noting that $\|E_0\| = 0$, we get :

$$\|E(x)\| \leqslant c_2 h^{r+1} \omega(h), \tag{30}$$

where $c_2 = \frac{2\|B\|\ e_{\|A\|}}{(r+2)!\,\|A\|}$ is a constant independent of h.

By definition (29), it follows that :

$$e(x) \leqslant c_2 h^{r+1} \omega(h) = 0(h^{\alpha+r+1}) \tag{31}$$

and

$$\bar{e}(x) \leqslant c_2 h^{r+1}\ \omega(h) = 0(h^{\alpha+r+1}). \tag{32}$$

We now estimate $|y'(x) - S_k'(x)|$. Thus, using (7), (9) and the Lipschitz conditions (3—4), we get :

$$e'(x) = |y'(x) - S_k'(x)| \leqslant c_3(e_k + \bar{e}_k) + 2L_1 \frac{h^{r+1}}{(r+1)!} \omega(h) \tag{33}$$

where $c_3 = L_1 + L_1^2(e - 1) + \frac{L_1^2}{(r+1)!} + L_1 L_2 (e - 1) + \frac{L_1 L_2}{(r+1)!}$ is a constant independent of h.

On using inequalities (31) and (32) into inequality (33), we easily get :

$$e'(x) \leqslant c_4 h^{r+1} \omega(h) = 0(h^{\alpha+r+1}) \tag{34}$$

where $c_4 = 2c_2 c_3 + \frac{2L_1}{(r+1)!}$ is a constant independent of h.

In a similar manner, we estimate $|z'(x) - \bar{S}_k'(x)|$.

From (8), (10) and using the Lipschitz conditions (3—4), we get :

$$\bar{e}'(x) = |z'(x) - \bar{S}'_k(x)| \leqslant c_5(e_k + \bar{e}_k) + 2L_2 \frac{h^{r+1}}{(r+1)!} \omega(h) \tag{35}$$

where $c_5 = L_2 + L_1 L_2 (e - 1) + \frac{L_1 L_2}{(r+1)!} + L_2^2 (e - 1) + \frac{L_2^2}{(r+1)!}$ is a constant independent of h.

Using inequalities (31) and (32), inequality (35) becomes :

$$\bar{e}'(x) \leqslant c_6 h^{r+1} \omega(h) = 0(h^{\alpha+r+1}) \tag{36}$$

where $c_6 = 2c_2 c_5 + \frac{2L_2}{(r+1)}$ is a constant independent of h.

Using (3), (4), (31), (32), (34) and (36) we easily get for all $i = 2(1)$ $r+1$.

$$|y^{(i)}(x) - S_k^{(i)}(x)| \leqslant c^* h^{r+1} \omega(h) = 0(h^{\alpha+r+1}) \tag{37}$$

and

$$|z^{(i)}(x) - \bar{S}_k^{(i)}(x)| \leqslant \bar{c}^* h^{r+1} \omega(h) = o(h^{\alpha+r+1}) \tag{38}$$

where c^* and $\bar{c}^*$ are constants independent of h.

Thus, we have proved the following

THEOREM. *Let $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ be the approximate solutions to problem (1)—(2) given by equations (7—8), and let $f_1, f_2 \in C^r([x_0, x_n] \times R^2)$. Then, for all $x \in [x_0, x_1]$ we have :*

$$|y(x) - S_0(x)| \leqslant \frac{2L_1}{(r+2)!} h^{r+2} \omega(h),$$

$$|y^{(j)}(x) - S_0^{(j)}(x)| \leqslant \frac{2L_1}{(r+1)!} h^{r+1} \omega(h), \quad j = 1(1)r \times 1$$

$$|z(x) - \bar{S}_0(x)| \leqslant \frac{2L_2}{(r+2)!} h^{r+2} \omega(h)$$

and

$$|z^{(j)}(x) - \bar{S}_0^{(j)}(x)| \leqslant \frac{2L_2}{(r+1)!} h^{r+1}\omega(h), \ j = 1(1)r+1$$

and for all $x \in [x_k, x_{k+1}]$, $k = 1(1)n - 1$ *we have :*

$$|y^{(j)}(x) - S_k^{(j)}(x)| \leqslant Ch^{r+1}\omega(h), \ j = 0(1)r+1$$

and

$$|z^{(j)}(x) - \bar{S}_k^{(j)}(x)| \leqslant Kh^{r+1}\omega(h), \ j = 0(1)r+1,$$

where C *and* K *are constants independent of* h.

Numerical example. Consider the following system of differential equations :

$$y' = y + z - x - x^2 - e^{2x},$$

$$z' = 2y + 2z - 2e^x - 2x^2 - 2, \ y(0) = 1, \ z(0) = 2.$$

The method is tested using this example in the interval [0,1] with step size $h = 0.1$ where $r = 0$ and 1.

The analytical solution is :

$$y(x) = e^x + x,$$

$$z(x) = e^{2x} + x^2 + 1.$$

The results tabulated below are evaluated at the point 0.25.

	Analytical value		Numerical value	Absolute error
y	1.534025417	$r=0$ $r=1$	1.530346203 1.533906117	0.003679214 0.0001193
z	2.711221271	$r=0$ $r=1$	2.70386284 2.710982672	0.007358431 0.000238599
y'	2.284025417	$r=0$ $r=1$	2.261907185 2.283397416	0.022118232 0.000628001
z'	3.797442543	$r=0$ $r=1$	3.753206079 3.796186541	0.044236464 0.001256002
y''	1.284025155	$r=1$	1.282951722	0.001073433
z''	8.594884559	$r=1$	8.59273769	0.002146869

REFERENCES

[1] M i c u l a , G h., *Spline functions of higher degree of approximation for solutions of systems of differential equations.* Studia Univ. Babeş-Bolyai, ser. Math-Mech., **17** (1972), fasc. 1, 21–32.

[2] M i c u l a , G., *Approximate integration of systems of differential equations by spline fu nctions.* Studia Univ. Babeş-Bolyai Cluj, series Mathematica, fasc. **2**, 1971, pp. 27–39.

[3] S c h u m a k e r , L a r r y L. *Optimal spline solutions of systems of ordinary differentia equations.* Differential equations (São Paolo ,1981), pp. 272–283, Lecture Notes in Math., 957, Springer, Berlin–New York, 1982.

[4] F a w z y T h., R a m a d a n Z., *Spline approximations for system of ordinary differential equations.* Annales Univ. Sci. Budapest, sec. comp., Vol. **7** (to appear).

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