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ON THE EXISTENCE OF LIMIT CYCLES FOR AN AUTONOMOUS SYSTEM

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The number of limit cycles of an autonomous system can be studied in the case when the system has one or more singular points. Papers [1], [2] gave the conditions that a system of the form

$$\begin{cases}\dot{x} = y - F(x)\\ \dot{y} = -g(x)\end{cases}$$

admits exactly n limit cycles, where the origin is the only singular point. In our papers [4] and [5] the above system is generalized, and the conditions that the system admits exactly n small amplitude limit cycles, the origin being the only singular point, are given. Denisov gives in [3] conditions for the existence of a limit cycle which surrounds three singular points, or an a odd number of singular points lying on the Ox-axis. There are also given existence conditions for two limit cycles surrounding three singular points. In [6] and [7] Zhilevich establishes conditons that an autonomous system of the form:

$$\begin{cases}\dot{x} = y - F(x)\\ \dot{y} = -g(x)\end{cases}$$

admits k limit cycles surrounding an odd number of singular points lying on the Ox-axis. We generalize further down the system studied by Zhilevich and give existence conditions for k limit cycles surrounding $2n+1$ singular points lying on the Ox-axis.

Let the system

$$(1)\quad \begin{cases}\dot{x} = yh(x) - F(x)\\ \dot{y} = -g(x)\end{cases}$$

be, where $F(x) = \int\limits_0^x f(x)\, dx$, $g(x)$, $f(x)$ and $h(x)$ are continuous functions.

(A) $g(\alpha_i) = 0,\ i = -1, 0, 1, \ldots, 2n-1;\ \alpha_{-1} < \alpha_0 < \alpha_1 < \ldots < \alpha_{2n-1}$

$g(x) > 0\ \forall x \in (\alpha_{2i-3}, \alpha_{2i-2}),\ i = \overline{1, n},\ x \in (\alpha_{2n-1}, \infty)$

$g(x) < 0,\ \forall x \in (\alpha_{2i-2}, \alpha_{2i-1}),\ i = \overline{1, n},\ x \in (-\infty, \alpha_{-1})$

Likewise, let the functions:

$$F_i(x) = F(x) - \varphi_i(x)\, h(x),\ \emptyset_i(x) = F(x) - \varphi_i(x)\ h(x) - \frac{g(x)}{\varphi_i'(x)}.$$

THEOREM 1. *Let the function $g(x)$ be which satisfies the conditions (A). There exist the functions $\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x)$, $k \geqslant 2$ and the systems of numbers:*

$$x_{-k} < x_{-k+1} < \ldots < x_{-1} < \alpha_{-1};\ x_k > x_{k-1} > \ldots > x_1 > \alpha_{2n-1}.$$

We impose the supplementary conditions:

(i) $\quad \varphi_i(0) = 0,\ \varphi_i'(x)\,(-1)^i < 0,\ i = \overline{1, k},\ x_{-i} \leqslant x \leqslant x_i,$

(ii) $\quad h(x_{-i})\,F_i(x_{-i})\,(-1)^{i+1} < \emptyset_i(x)\,(-1)^{i+1} < h(x_i)\,F_i(x_i)\,(-1)^{i+1},$

$$h(x) > 0 \qquad x \in [x_{-i},\ x_i]$$

and

$$h(x_{-1})\,F_i(x_{-i})\,(-1)^{i+1} \leqslant (-1)^{i+1}\, h(x)\,F_i(x) \leqslant h(x_i)\,F_i(x_i)\,(-1)^{i+1},$$

$$x \in [x_{-i},\ x_i],\ i = \overline{1, k}.$$

(iii) $\quad h(x_i)\,F_i(x_i)\,(-1)^{i+1} > (-1)^{i+1}\,F_i(-_{i+1})\,x(h_{-i+1})$

$$h(x_{-i})\,F_i(x_{-i})\,(-1)^i > (-1)^i\,F_i(x_{i-1})\,h(x_{i-1}).$$

Then, the system (1) *has $k - 1$ limit cycles sorrounding $2n + 1$ singular points. In every domain $x_{-i} < x < x_i$, $i = \overline{1, k}$, there are at most $i - 1$ limit cycles, out of which $[i/2]$ are stable and $[(i-1)/2]$ are unstable.*

Proof. Let $k = 2$. We substitute in (1):

$$\bar{y} = y + \varphi_1(x). \tag{2}$$

Then the system (1) becomes:

$$\begin{cases} \dot{x} = \bar{y}h(x) - F_1(x) \\ \dot{\bar{y}} = -\ \varphi_1'(x)\ [\bar{y}h(x) - \emptyset_1(x)] \end{cases} \tag{3}$$

We build a rectangle Γ_1 having the legs parallel to the coordinate axes, and the vertices $B_1(x_1, h(x_1)F_1(x_1))$, $B_{-1}(x_{-1},\ h(x_{-1})\,F_1(x_{-1}))$. From the conditions of the theorem it results that the trajectories of the system (3) which cross the rectangle legs for increasing t penetrate inside the rectangle. Then, we substitute in (3):

$$\bar{\bar{y}} = y + \varphi_1(x) - \varphi_2(x), \tag{4}$$

obtaining the system:

$$\begin{aligned} \dot{x} &= h(x)\,\bar{\bar{y}} - F_2(x) \\ \dot{\bar{\bar{y}}} &= -\ \varphi_2'(x)\ [\bar{\bar{y}}h(x) - \emptyset_2(n)]. \end{aligned} \tag{5}$$

Let $\overline{\Gamma}_1$ be the closed curve into which the border of the rectangle Γ_1 passes through the transformation (4). The trajectories of the system (5) crossing the curve $\overline{\Gamma}_1$ penetrate inside it. Then we build the rectangle Γ_2 having the legs parallel to the coordinate axes, and the vertices $B_2(x_2, h(x_2)\ F_2(x_2))$, $B_{-2}(x_{-2},\ h(x_{-2})F_2(x_{-2}))$. The trajectories of the system (5) crossing the legs of Γ_2 go out of the rectangle.

Then, in the ring domain bounded by Γ_2 and $\overline{\Gamma}_1$ there exist at least one unstable limit cycle, therefore the theorem is proved for $k = 2$. For $k \geqslant 3$, the proof is analogous.

Notice. If the conditions (i)—(iii) are adequately modified, then the limit cycle in the case $k = 2$ is stable.

THEOREM 2. *If the conditions of Theorem 1 hold, and if we impose the supplementary conditions:*

$(1a)$ $\quad f(x) < 0,\ \forall x \in (\beta_{-1}, \beta_1);\ f(\beta_{-1}) = f(\beta_1) = 0,\ \beta_1 < \alpha_{-1},$

$$\beta_2 > \alpha_{2n-1}.$$

$(1b)$ $\quad F(\alpha_i) = g(\alpha_i),\ i = -1, 0, 1, \ldots, 2n-1;\ F(\beta_{-1}) = F(\beta_1) = 0,$

$$\beta_{-1} < \alpha_{-1},$$

$$\beta_1 > \alpha_{2n-1};\ F(x)\,g(x)/h(x) < 0,\ \forall x \in (\beta_{-1}, \beta_1),\ x \neq \alpha_i,$$

$$i = -1, 0, 1, \ldots, 2n-1,$$

and, additionally,

$$\int_0^{\beta_i} \frac{g(x)}{h(x)}\, dx > 0,\ i = -1, 1;\ \int_0^t \frac{g(x)}{h(x)}\, dx \leqslant \min_i \int_0^{\beta_i} \frac{g(x)}{h(x)}\, dx, \tag{2}$$

$$t \in (\alpha_1,\ \alpha_{2n-2}),$$

then system (1) has at least k limit cycles surrounding $2n + 1$ singular points, since in every domain $x_{-1} < x < x_i$ there are at least i limit cycles, out of which $[(i+1)/2]$ are stable, and $[i/2]$ are unstable.

Proof. The proof is analogous to that in the case of Theorem 1, the difference consisting of the fact that as the first border we consider the curve:

$$c_0 = \frac{(y + \varphi_1)^2}{2} + \int_0^x \frac{g(s)}{h(s)}\, ds,$$

where $c_0 = \min\limits_i \int_0^{\beta_i} \frac{g(s)}{h(s)}\, ds,\ i = -1, 1.$

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