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I-POINTS WITH RESPECT TO A GIVEN SET

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Let $(V, \| \ \|)$ be a normed space and let M, X be two nonvoid subsets of V.

DEFINITION 1. *We say that a point $x \in X \cap M$ has the I-property with respect to M, if there is an $r > 0$ such that $S(x, r) \cap M \subseteq X$, where $S(x, r) = \{y \in V \mid \|x - y\| < r\}$.*

Let $I_M(X) = \{x \in X \cap M \mid x$ has the I-property$\}$.

Remark 1. It is easy to see that the following assertions are true:

(a) $M \cap \operatorname{int} X \subseteq I_M(X) \subseteq M \cap X$;

(b) if $M = V$, then $I_M(X) = \operatorname{int} X$;

(c) ext $X \cap I_M(X) = \Phi$;

(d) $I_M(M) = M$.

DEFINITION 2. *We say that the set X has the I-property with respect to the set M if $X = \Phi$ or if $I_M(X) = X \cap M$.*

Example 1. *Let $V = R$, $M =$ the set* of rational numbers and $X = \{0, 1, 3\}$. We have $I_M(X) = \Phi$.

Example 2. Let $V = R$, $M =$ the set of rational numbers and $X = \{x \in R \mid -\sqrt{2} \leqslant x \leqslant \sqrt{2}\}$. We have $I_M(X) = M \cap X$.

Example 3. Let $V = R$, $M =$ the set of integers numbers and $X = \{0, 1, 3\}$. We have $I_M(X) = X$.

PROPOSITION 1. *If A_i $i = 1, \ldots, m$ are subset of V with the I-property, then $\bigcap_{i=1}^{m} A_i$ has the I-property with respect to M.*

Proof. Let $z \in M \cap \left(\bigcap_{i=1}^{m} A_i\right)$. Because A_i $i = 1, \ldots, m$ has the I-property with respect to M, there are $r_i > 0$, $i = 1, \ldots, m$ such that $S(z, r_i) \cap M \subseteq A_i$ $i = 1, \ldots, m$. Let $r = \min\ \{r_i \mid i = 1, \ldots, m\}$. Because $S(z, r) \subseteq S(z, r_i)$ for all $i \in \{1, \ldots, m\}$, we have

$$S(z, r) \cap M \subseteq \bigcap_{i=1}^{m} A_i.$$

Hence $z \in I_M(X)$. Since z was arbitrarily chosen, it follows that $M \cap \left(\bigcap_{i=1}^{m} A_i\right) = I_M\left(\bigcap_{i=1}^{m} A_i\right)$.

PROPOSITION 2. *If A_i, $i \in I$ is a family of subsets of V with the I-property with respect to M, then $\bigcup_{i\in I} A_i$ has the I-property with respect to M.*

Proof. Let $z \in \bigcup_{i\in I} A_i$. Then there is a $j \in I$ such that $z \in A_j$. Because A_j has the I-property with respect to M, there is an $r > 0$ such that $S(z, r) \cap M \subseteq A_j$. Because $A_j \subseteq \bigcup_{i\in I} A_i$, it follows that $S(z, r) \cap M \subseteq \bigcup_{i\in I} A_i$. Hence $z \in I_M(\bigcup_{i\in I} A_i)$. Since z was arbitrarily chosen, it results that $M \cap (\bigcup_{i\in I} A_i) = I_M(\bigcup_{i\in I} A_i)$.

Let $\mathscr{I} = \{A \subseteq V \mid A$ has the I-property with respect to $M\}$. From definition 2, remark 1 and propositions 1 and 2 it follows that the family $\mathscr{I}$ is a topology on V.

DEFINITION 3. *We say that a point $x \in X$ has the c-property if there is an $r > 0$ such that $S(x, r) \subseteq \mathrm{conv}X$, where $\mathrm{conv}X$ denotes the convex hull of X.*

Example 4. Let $X = \{(0,0), (1,2), (2,1), (1,1)\}$. The point $(1,1)$ has the c-property, but the points $(0,0)$, $(1,2)$ and $(2,1)$ do not have the c-property.

Remark 2. If $x \in \mathrm{int}X$, then x has the c-property.

DEFINITION 3. *We say that the set X has the c-property if all the points of X have the c-property.*

DEFINITION 4. *We say that the set X has the c-property with respect to M if all the points of $X \cap M$ have the c-property.*

Example 5. Let Z be the set of all integer numbers and let $X = \{(x, y) \in R^2 \mid x \geqq 1/2,\ y \geqq 1/2,\ 2x + 2y \leqq 5\}$. The set X has the c-property with respect to $M = Z^2$.

PROPOSITION 3. *If A_i, $i \in I$ is a family of subsets of V with the c-property with respect to M, then $\bigcup_{i\in I} A_i$ has the c-property with respect to M.*

Proof. Let $z \in M \cap (\bigcup_{i\in I} A_i)$. Then there is a $j \in I$ such that $z \in A_j$. Because A_j has the c-property with respect to M, there is an $r > 0$ such that $S(z, r) \subseteq \mathrm{conv}A$. But $\mathrm{conv}A_j \subseteq \mathrm{conv}(\bigcup_{i\in I} A_i)$. Then $S(z, r) \subseteq \mathrm{conv}(\bigcup_{i\in I} A_i)$. Hence z has the c-property. Because z was arbitrarily chosen, it follows that the set $\bigcup_{i\in I} A_i$ has the c-property with respect to M.

Let

$c - I(X) = \{x \in X \cap M \mid \exists\, r > 0$ such that $S(x, r) \subseteq \mathrm{conv}X$ and $x \in I_M(X)\}$.

PROPOSITION 4. *If X is a strongly convex set with respect to M, then the set $c - I_M(X)$ is also strongly convex with respect to M.*

Proof. Let $x \in M \cap \mathrm{conv}(c - I_M(X))$. Then there is a natural number m and there are $x^1, \ldots, x^m$ elements of $c - I_M(X)$ and $t_1 \geqslant 0, \ldots, t_m \geqslant 0$ with $t_1 + \ldots + t_m = 1$ such that $x = t_1 x^1 + \ldots + t_m x^m$. For each $i \in \{1, \ldots, m\}$ there is a positive number r such that $S(x^i, r_i) \subseteq \mathrm{conv}X$.

Let $r = \min\{r_i \mid i = 1, \ldots, m\}$, and let $z \in S(x, r)$. For each $j \in \{1, \ldots, m\}$, consider the point

$$z^j = z + \sum_{i=1}^{m} t_i (x^j - x^i).$$

For each $j \in \{1, \ldots, m\}$, we have $\| x^j - z^j \| = \| x - z \| < r \leqslant r_j$. Then $z^j \in S(x^j, r_j)$ for all $j \in \{1, \ldots, m\}$. Because $z = \sum_{i=1}^{m} t_i z^i$ and $S(x^i, r_i) \subseteq \mathrm{conv}X$ for all $i \in \{1, \ldots, m\}$, we get that $z \in \mathrm{conv}X$. But z was arbitrarily chosen. Hence $S(x, r) \subseteq \mathrm{conv}X$.

From the definition of the strongly convex set with respect to a given set (see [1]) we get that $M \cap \mathrm{conv}X \subseteq X$. Then we have

$$S(x, r) \cap M \subseteq M \cap \mathrm{conv}X \subseteq X.$$

Hence $x \in c - I_M(X)$.

Because x was arbitrarily chosen in $M \cap \mathrm{conv}(c - I_M(X))$, we get that the $M \cap \mathrm{conv}(c - I_M(X)) \subseteq c - I_M(X)$. Hence $c - I_M(X)$ is strongly convex with respect to M.

REFERENCES

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