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# ON AN INEQUALITY OF NANSON

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In this paper we give a generalization of the following inequality of Nanson [1] (see also [2]) for the case of p,q-convex sequences :

THEOREM 1. *If the real sequence* $a=(a_1,\ldots,a_{2n+1})$ *is convex, then*

$$\frac{a_1+a_3+\ldots+a_{2n+1}}{n+1}\geqslant\frac{a_2+a_4+\ldots+a_{2n}}{n} \tag{1}$$

*with equality if and only if a presents an arithmetic progression.*

*Definition. A real sequence* $a=(a_1, a_2,\ldots)$ *is* $p,\ q$*-convex* $(p,\ q>0)$ *if* $L_{pq}(a_n)\geqslant 0$ *for* $n\geqslant 1$, *where*

$$L_{pq}(a_n)=a_{n+2}-(p+q)a_{n+1}+pqa_n. \tag{2}$$

*Remark* 1. For $p=q=1$ we get the usual notion of convexity. In [4] we have shown that a remarkable place in the theory of $p,\ q$-convex sequences is played by the sequence $w=(w_1,\ w_2,\ldots)$ given by

$$w_n=\begin{cases}\dfrac{p^n-q^n}{p-q} & \text{if } p\neq q\\ np^{n-1} & \text{if } p=q.\end{cases} \tag{3}$$

For example, we have proved :

LEMMA 1. *The sequence a satisfies the relation*

$$L_{pq}(a_n)=0,\ n=1,\ldots \tag{4}$$

*if and only if*

$$a_n=uw_n+vw_{n+1} \tag{5}$$

*where u and v are arbitrary real numbers.*

THEOREM 2. *If the real sequence* $a=(a_1,a_2,\ldots,a_{2n+1})$ *is* $p,\ q$*-convex, then :*

$$\frac{(pq)^na_1+(pq)^{n-1}a_3+\ldots+a_{2n+1}}{w_{n+1}}\geqslant\frac{(pq)^{n-1}a_2+(pq)^{n-2}a_4+\ldots+a_{2n}}{w_n} \tag{6}$$

*with equality if and only if a satisfies* (4).

*Proof.* Since

$$a_{2k+1} - (p+q)a_{2k} + pqa_{2k-1} \geqslant 0, \ k = 1,\ldots,n$$

and

$$a_{2k+2} - (p+q)a_{2k+1} + pqa_{2k} \geqslant 0, \ k = 1,\ldots,n-1,$$

we have the inequalities:

$$\sum_{k=1}^{n} \frac{w_k w_{n-k+1}}{(pq)^{k-1}} (a_{2k+1} - (p+q)a_{2k} + pqa_{2k-1}) \geqslant 0 \tag{7}$$

and

$$\sum_{k=1}^{n-1} \frac{w_k w_{n-k}}{(pq)^{k-1}} (a_{2k+2} - (p+q)a_{2k+1} + pqa_{2k}) \geqslant 0. \tag{8}$$

Because:

$$w_{k+1}w_{n-k} - pqw_k w_{n-k-1} = w_n \tag{9}$$

adding (7) and (8) we obtain:

$$pqw_n \sum_{k=1}^{n+1} \frac{a_{2k-1}}{(pq)^{k-1}} - w_{n+1} \sum_{k=1}^{n} \frac{a_{2k}}{(pq)^{k-1}} \geqslant 0$$

which is (6).

*Remark* 2. For

$$s_n = w_1 + \ldots + w_n \tag{10}$$

we have

$$L_{pq}(s_n) = 1. \tag{11}$$

So we obtain the following:

LEMMA 2. *If the real sequence $a = (a_1, a_2, \ldots)$ satisfies:*

$$m \leqslant L_{pq}(a_n) \leqslant M, \ n = 1,2,\ldots \tag{12}$$

*then, the sequences $(b_n)$ and $(c_n)$, $n = 1,2,\ldots$, given by*

$$b_n = Ms_n - a_n; \ c_n = a_n - ms_n, \tag{13}$$

*are p,q-convex.*

THEOREM 3. *If the sequence a satisfies* (12), *then*

$$mz_n \leqslant \frac{(pq)^n a_1 + \ldots + a_{2n+1}}{w_{n+1}} - \frac{(pq)^{n-1}a_2 + \ldots + a_{2n}}{w_n} \leqslant M z_n \tag{14}$$

*where*

$$z_n = \frac{w_n + \sum_{k=1}^{n-1} (pq)^{n-k} w_k (w_{n-k+1} + w_{n-k})}{w_n w_{n+k}} \tag{15}$$



*Proof.* Because $(b_n)$ and $(c_n)$, $n = 1,2,\ldots$, given by (13), are *p,q*-convex, we may apply (6). Taking into account (11), we can make with $(s_k)$, $k = 1,2,\ldots$, the same operations as in the proof of Theorem 2 and we get (14) and (15).

CCOROLARY. *Since for $p = q = 1$, $w_n = n$,* (6) *becomes* (1) *and* (14) *an inequality proved in* [3], *where* $z_n = \dfrac{2n+1}{6}$.

## REFERENCES

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