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RELATIVELY DENSE UNIVERSAL SEQUENCES FOR THE CLASS OF CONTINUOUS PERIODICAL FUNCTIONS OF PERIOD T

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1. Introduction. It is well known that if $P \in Q[X]$ is a polynomial with rational coefficients, then the sequence $(a_n)_{n\geqslant 1}$, $a_n = \sin P(n)$ is not convergent at 0 (see [2] pp. 145–146).

In [1] it was proved that if $f: \mathbb{R} \to \mathbb{R}$ is a continuous periodical function with irrational period T, then the sequence $(f(n))_{n\geqslant 1}$ is everywhere dense in the closed interval $[m, M]$, where m and M are the bounds of f.

In what follows, we shall say that a sequence of real numbers $(a_n)_{n\geqslant 1}$ is relatively dense for the function $f: \mathbb{R} \to \mathbb{R}$ if, for every $x, y \in R$ with $f(x) < f(y)$, there exists an $n \geqslant 1$ such that $f(x) < f(a_n) < f(y)$. If f is a continuous bounded function which attains its bounds m and M, it is clear that $(a_n)_{n\geqslant 1}$ is a relatively dense sequence for f if and only if the sequence $(f(a_n))_{n\geqslant 1}$ is everywhere dense in $[m, M]$. Then, it is natural to ask if on can find a sequence $(a_n)_{n\geqslant 1}$ of real numbers which is relatively dense for every continuous periodical function $f: R \to R$ of period. T. We shall call a sequence with this property a "relatively dense universal sequence" for the class of continuous periodical functions of period T.

The result of [1] shows that if T is an irrational number, then the sequence $(a_n)_{n\geqslant 1}$, $a_n = n$, is a relatively dense universal sequence for the class of continuous periodical functions $f: R \to R$ of period T.

In this paper, we shall construct four classes of relatively dense universal sequences for the set of continuous periodical functions $f: R \to R$ of period T.

In the second part of the paper, it is proved that if $(a_n)_{n\geqslant 1}$ is an unbounded sequence of positive numbers with the property $\limsup_{n\to\infty} (a_{n+1} - a_n) = 0$, then it is a relatively dense universal sequence for the class of continuous periodical functions $f: R \to R$ of period T.

In the third part of the paper, we prove that if $P \in R[X]$ has the property $\frac{1}{T}(P - a_0) \notin Q[X]$, $T \in R\setminus\{0\}$, then $(P(n))_{n\geqslant 1}$ is a relatively dense universal sequence for the class of continuous periodical functions $f: R \to R$ of period T. This result is much better than those of [2] and [1].

In the fourth part of the paper, based on the results of H. FURSTENBERG [3], we give other two classes of relatively dense universal sequences for the continuous periodical functions of period T.

First, we shall give some definitions.

1.1. DEFINITION. *Let $J \subset R$ be an interval (finite or not). We shall say that the set A is everywhere dense in J if $A \subseteq J$ and, for every open nonempty interval I, we have $(I \cap J) \cap A \neq \emptyset$.*

1.2. DEFINITION. *Let $(u_n)_{n \geq 1}$ be a sequence of real numbers and let $r > 0$. We shall say that this sequence is uniformly distributed* (mod r) *if for every $0 \leq \alpha < \beta \leq r$, we have*

$$\lim_{n\to\infty} \frac{v_n(\alpha, \beta)}{n} = \frac{\beta - \alpha}{r},$$

where $v_n(\alpha, \beta) = \text{card}\left\{k : 1 \leq k \leq n, u_k - r\left[\frac{u_k}{r}\right] \in (\alpha, \beta)\right\}$.

1.3. REMARK. From definition 1.2, it follows directly that the sequence $(u_n)_{n\geq 1}$ is uniformly distributed (mod 1) if and only if the sequence $(v_n)_{n\geq 1}$, given by $v_n = r u_n$, is uniformly distributed (mod r).

2. The first class of relatively dense universal sequences for the continuous periodical functions of period T

2.1. THEOREM. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of positive real numbers. If*

α) $(a_n)_{n\geq 1}$ is unbounded

β) $(b_n)_{n\geq 1}$ is unbounded

γ) $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$,

then the set $M = \left\{\frac{a_n}{b_m} : m, n \in N\setminus\{0\}\right\}$ is everywhere dense in the interval $[0, \infty)$.

Proof. We show that $[0, \infty) \subset \overline{M}$. Let $Z \in [0, \infty)$ and let V be a neighbourhood of Z in R. Then there exists an interval $[x, y) \subset V$, where $x \leq Z < y$

When $Z = 0$, we have $x = Z = 0$ and from condition β), it follows that there exists an $m \in N$ with $b_m > a_1/y$ such that $a_1/b_m \in M \cap V$.

When $Z > 0$, we suppose that $0 < x < Z$. Let $(x - y)/x > 0$. From γ), it follows that there exists an $n_0 \in N$ such that:

$$\frac{a_n}{a_{n-1}} < 1 + \varepsilon \text{ for every } n > n_0. \tag{2.1}$$

From β), it follows that there exists an $m \in N$ with $b_m \geq a_{n_0}/x$. If the number n is given by $n = \inf\{p \in N : p > n_0, a_p > x \cdot b_m\}$, then it satisfies

$$\frac{a_{n-1}}{x} \leq b_m < \frac{a_n}{x}. \tag{2.2}$$

Then, for $n > n_0$, by (2.1), one obtains

$$x\frac{a_n}{a_{n-1}} < x + \varepsilon x = y. \tag{2.3}$$

From (2.2) and (2.3), we get

$$x < \frac{a_n}{b_m} \leq x\frac{a_n}{a_{n-1}} < y$$

so that $a_n/b_m \in M \cap (x, y) \subset M \cap V$.

In both cases, we get $Z \in \overline{M}$ and Theorem 2.1 is proven.

2.2. REMARK. The folllowing examples show that without one of the conditions α), β) or γ), Theorem 2.1 is not generally true.

a) $a_n = 1$, $b_n = n$

b) $a_n = n$, $b_n = 1$

c) $a_n = b_n = c^n$, where $c > 1$. In the last example, we have $\frac{1}{2}(c + c^2) \notin M$.

2.3. COROLLARY. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of positive real numbers with the properties:*

α) $(a_n)_{n\geq 1}$ is unbounded

β) $(b_n)_{n\geq 1}$ is unbounded

γ) $\limsup\limits_{n\to\infty} (a_{n+1} - a_n) = 0$.

Then the set $M = \{a_n - b_m : n, m \in N\setminus\{0\}\}$ is everywhere dense in R.

Proof. Take $A_n = e^{a_n}$, $B_n = e^{b_n}$ for $n \geq 1$. It can be easily seen that the sequences $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ satisfy the conditions of Theorem 2.1. Therefore, the set $M_1 = \{A_n/B_m : n, m \in N\setminus\{0\}\} = \{e^{a_n - b_m} : n, m \in N\setminus\{0\}\}$ is everywhere dense in the interval $[0, \infty)$. It follows that the set M is everywhere dense in R.

If the sequences of positive numbers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are increasing, then we regain the result of M. SOMOS [7].

2.4. THEOREM. *An unbounded sequence $(a_n)_{n\geq 1}$ of positive real numbers having the property $\limsup\limits_{n\to\infty} (a_{n+1} - a_n) = 0$ is a relatively dense universal sequence for the class of all continuous periodical functions $f : R \to R$.*

Proof. Using Corollary 2.3, we get that the set $M = \{a_n - mT : n, m \in N\setminus\{0\}\}$ is everywhere dense in R.

Let $f : R \to R$ be a continuous periodical function with period T and let $y \in [m, M]$. Then, there exists an $x \in R$ such that $f(x) = y$. As M is everywhere dense in R, it follows that there exist two sequences of natural numbers $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ such that $\lim\limits_{k\to\infty} (a_{n_k} - m_k T) = x$.

Then, by the continuity of f, we get

$$\lim_{k\to\infty} f(a_{n_k}) = \lim_{k\to\infty} f(a_{n_k} - m_k T) = f(x) = y$$

and the theorem is proven.

2.5. COROLLARY. *Suppose that the sequences of positive real numbers* $(a_n^{(i)})_{n\geq 1}$ *for* $i=\overline{1, q}$ *are unbounded and that* $\limsup_{n\to\infty} (a_{n+1}^{(i)} - a_n^{(i)}) = 0$ *for all* $i = \overline{1, q}$. *Then the sequence* $(A_n)_{n\geq 1}$ *defined by* $A_n = \sum_{i=1}^{q} \gamma_i a_n^{(i)}$, *where* $\gamma_i > 0$ *for* $i = \overline{1, q}$ *is a relatively dense universal sequence for the class of all continuous periodical functions* $f: R \to R$.

The proof is an immediate consequence of Theorem 2.4.

2.6. APPLICATIONS. (*i*) The sequence $(P(n))_{n\geq 1}$, where P is a „generalized polynomial" $P(X) = \gamma_1 X^{\alpha_1} + \ldots + \gamma_q X^{\alpha_q}$ with $\gamma_i > 0$, $\alpha_i \in (0, 1)$ for $i = \overline{1, q}$ is a relatively dense universal sequence for the class of all continuous periodical functions. This is equivalent to the assertion that the set $\{f(\gamma_1 n^{\alpha_1} + \ldots + \gamma_q n^{\alpha_q}) : n \in N\}$ is everywhere dense in $[m, M]$, where f is a continuous periodical function.

Thus, the sets $\{\sin (3\sqrt[3]{n} + \sqrt{n}) : n \in N\}$, $\{\cos (2\sqrt[4]{n} + 5\sqrt[7]{n}) : n \in N\}$ are everywhere dense in the interval $[-1,1]$.

(ii) The sequence $(\ln n)_{n\geq 1}$ is a relatively dense universal sequence for the class of all continuous periodical functions. Then the set $\{f(\ln n) : n \in N\setminus\{0\}\}$ is everywhere dense in $[m, M]$ for every continuous periodical function f. For example, the sets $\{\sin (\ln n) : n \in N\setminus\{0\}\}$, $\{\cos (\ln n) : n \in N\setminus\{0\}\}$ are everywhere dense in $[-1,1]$.

3. The second class of relatively dense universal sequence for the continuous periodical functions of period T

3.1. DEFINITION. *Let* $f: R \to R$ *and let* $P \in R[X]$, $T \in R\setminus\{0\}$. *We shall say that* f *is T-relatively periodical with respect to* P *if* $f(P(n) + mT) = f(P(n))$ *for all* $n \in N$ *and* $m \in Z$.

Apparently, this class of functions is larger than the class of periodical functions with the period T; but we shall see that, for example, for $P \in Q[X]$, T irrational and f continuous, they coincide.

We shall now present some lemmas to be used in the proofs, and which are self-important.

3.2. LEMMA. *Let* $s > 0$, $a \geq 0$, $b \geq 0$ *be integer numbers and let* θ *be an irrational number. Then the set*

$$A = \{n + m\theta : m \in N,\ n \in Z,\ n = a \pmod s),\ m = b \pmod s)\}$$

is everywhere dense in R.

Before giving the proof, we notice that the lemma is well known in the case $s = 1$ and $a = b = 0$. The above generalization is of the same type as that from [4] (where it appears in a problem of diophantine approximation).

Proof. Let

$$B = \{a + m\theta : m \in N,\ m = b \pmod s)\} = \{a + b\,\theta + ks\,\theta : k \in N\},$$

Obviously, it will be sufficient to prove that the set

$$B \bmod s = \{x \bmod s : x \in B\} = \left\{x - s\left[\frac{x}{s}\right] : x \in B\right\}$$

is everywhere dense in the interval $[0, s)$.

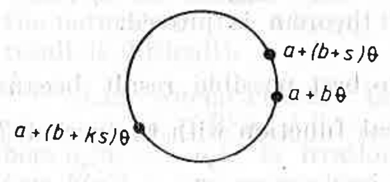

Fig. 1

Suppose that the interval $[0, s)$ was wrapped up on the circle of radius $s/2\pi$. The set $B \bmod s$ is obtained starting from the point $a + b\theta$ with the "step" $\theta_1 = s\theta$. As θ_1 is an irrational number, it follows that the points $\theta_1, \theta_2, \ldots$ are pairwise distinct. Among the points $\theta_1, \theta_2, \ldots, \theta_{sn+1}$, there are two between which the distance is smaller than $s/s.n = 1/n$. But n can be made as great as we want and it follows that $B \bmod s$ is everywhere dense in the interval $[0, 1)$ (if in k steps we go through the distance d on the circle, then evidently this is the distance which we go through in k steps starting from an arbitrary point.)

We shall see that, from Lemma 3.4, it follows that the set B is uniformly distributed (mod s).

The following lemma is one of the central results in [8].

3.3. LEMMA. (Wely [8], Satz 9). *Let* $P(X) = a_p X^p + \ldots + a_1 X + a_0 \in R[X]$ *be a polynomial such that at least one of the coefficients* $a_p, \ldots, a_1$ *is an irrational number. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} e^{2\pi i P(n)} = 0.$$

The proof of this lemma is elementary but it requires a long introductive argument which we omit herein.

From the foregoing lemma and from the well-known theorem on uniform approximation of integrable functions by trigonometric polynomials, there follows (see [5], vol. I, Abs. II, Chap., 4, § 2, ex. 162–172)

3.4. LEMMA. *Let* $P(X) = a_p X^p + \ldots + a_1 X + a_0 \in R[X]$ *be a polynomial such that at least one of the coefficients* $a_p, \ldots, a_1$ *is an irrational number. Then the sequence* $(P(n))_{n\geq 1}$ *is uniformly distributed* (mod 1)

3.5. *Let* $T \neq 0$ *be a real number and let* $P \in R[X]$, $P(X) = a_p X^p + \ldots a_1 X + a_0$ *be a polynomial such that* $\frac{1}{T}(P - a_0) \notin Q[X]$. *Then the sequence* $(P(n))_{n\geq 1}$ *is a relatively dense universal sequence for the continuous periodical function of the period* T.

Proof. As $P_1 = \frac{1}{T}(P - a_0) \notin Q[X]$, it follows, by Lemma 3.4, that the sequence $(X_n)_{n\geq 1}$ $X_n = P_1(n) = P(n)/T - a_0/T$ is uniformly distributed (mod 1). Whence the set $A = \{P(n) + mT : n \in N,\ m \in Z\}$ is everywhere dense in R.

Let $f : R \to R$ be a continuous periodical function with the period T and let $y \in [m, M]$ and $x \in R$ be such that $f(x) = y$. There are two sequences $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ with $n_k \in N$, $m_k \in Z$, $k = 1, 2, \ldots$ such that $\lim_{n\to\infty} (P(n_k) + m_kT) = x$. The function f being continuous and periodical with the period T, it follows that $y = f(x) = f(\lim_{k\to\infty} (P(n_k) + m_kT)) = \lim_{k\to\infty} f(P(n_k) + m_kT) = \lim_{k\to\infty} f(P(n_k))$, and theorem is proved.

3.6. Remark. This theorem gives the best possible result because for $P_1 = \frac{1}{T}(P - a_0) \in Q[X]$ and f periodical function with the period T, we evidently have

$$\text{card } \{f(P(n)) : n \in N\} < \infty.$$

The following corollary is contained in the proof of the last Theorem but we prefer to state it separately because it is self-important and gives a wide generalization for Lemma 3.2.

3.7. Corollary. *Let T be a real number, $T \neq 0$, and let $P(X) = a_pX^p + \ldots + a_1X + a_0 \in R[X]$ be such that $\frac{1}{T}(P - a_0) \notin Q[X]$. Then the set $A = \{P(n) + mT : n \in N, m \in Z\}$ is everywhere dense in R. The sequence $(P(n))_{n\geq 1}$ is uniformly distributed* (mod T).

3.8. Remark. It is interesting to find the polynomials $Q \in Z[X]$ differing from $X + k$, $k \in Z$ for which the set $B = \{P(n) + Q(m)T : n \in N, m \in Z\}$ is everywhere dense in R. A sufficient condition is that $Z \subseteq Q(Z)$ but it is easy to see that in this case grad $(Q) = 1$ and $Q = X + k$.

Using corollary 3.7, we can now prove

3.9. Theorem. *Let $T \neq 0$ be a real number, let $P(X) = a_pX^p + \ldots + a_1X + a_0 \in R[X]$ and let $f : R \to R$ be relatively T-periodical continuous with respect to function P. If $P_1 = \frac{1}{T}(P - a_0) \notin Q[X]$, then f is a periodical function with the period T.*

Proof. Let $x \in R$. From Corollary 3.7, it follows that there exist the sequences $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ with $n_k \in N$, $m_k \in Z$, $k = 1, 2, \ldots$ such that $\lim_{k\to\infty} (P(n_k) + m_kT) = x$.

Then

$$f(x) = f(\lim_{k\to\infty} (P(n_k) + m_kT)) = \lim_{k\to\infty} f(P(n_k) + m_kT) =$$
$$= \lim_{k\to\infty} f(P(n_k)) = \lim_{k\to\infty} f(P(n_k) + (m_k + 1)T) =$$
$$= \lim_{k\to\infty} f\, P(n_k) + m_kT + T) = f(\lim_{k\to\infty} (P(n_k) + m_kT) + T) = f(x + T).$$

3.10. Remark. Theorem 3.9 shows that it is meaningless to state Theorem 3.5 for a weaker assumption and, evidently, it is sufficient for the proof that f be relatively T-periodical with respect to P.

3.11. G. RHIN proved in [6], by generalizing the results of I. M. VINOGRADOV, that lemmas 3.3 and 3.4 (which are equivalents) remain true if in their statements we replace everywhere the natural number n by p_n, the n-th prime number.

Using this result, it follows that Theorem 3.5, Corollary 3.7 and Theorem 3.9 remain valid if in their statements we replace everywhere the natural number n by p_n, the n-th prime number. (The proof of RIHN's result is difficult).

3.12. Applications. Let $P = a_pX^{r_p} + \ldots + a_1X^{r_1} + a_0$, where $a_p, \ldots, a_1, a_0 \in R$, $a_p \neq 0$, $r_p, \ldots, r_1 \in Q \setminus \{0\}$ and at least one of the numbers $a_p/\pi, \ldots, a_1/\pi$ is irrational. Then: (i) the sequences $(\sin P(n))_{n\geq 1}$, $(\cos P(n))_{n\geq 1}$ are everywhere dense in the interval $[-1,1]$.

(*ii*) the sequence $(\text{tg } P(n))_{n\geq 1}$ is everywhere dense in R. To prove this, it is sufficient to notice that if q is the least common denominator of the numbers $r_1, \ldots, r_p \in Q \setminus \{0\}$, then $P(X^q) \in Q[X]$ and so we can apply Theorem 3.5.

3.13. Remark. Applications 2.6. and 3.12 leave open the problem of density in $[-1,1]$ of the sequence $(\sin P(n))_{n\geq 1}$, where $P = a_pX^{\alpha_p} + \ldots + a_1X^{\alpha_1} + a_0$ is a "generalized polynomial" with $a_p, \ldots, a_1, a_0 \in R$, $a_p \neq 0$, $\alpha_i > 1$ and at least one α_i is irrational.

Generally can sufficient conditions be found for a "generalized polynomial" P such that the sequence $(f(P(n))_{n\geq 1}$ be everywhere dense in $[m, M]$ for every $f : R \to R$ continuous and periodical with the period T?

Taking into account the previous results, we make the conjecture that, at least in the first case above, the result of applications 2.6 and 3.12 remains true.

4. Completions and remarks.

We shall briefly present the results of H. FURSTENBERG [3] and using them, we shall construct new classes of sequences to which the result of the foregoing sections can be applied.

4.1. Definition. *A multiplicative semigroup $\Sigma \subseteq Z$ is lacunal if the set $\Sigma^+ = \{\sigma \in \Sigma : \sigma \geq 0\}$ contains only the powers of the positive integer a. Otherwise, the semigroup is nonlacunal.*

For example,

$\Sigma_1 = \{a^p, a^{p+1}, \ldots, a^{p+n}, \ldots\}$ is a lacunal semigroup for every $a \in N \setminus \{0\}$ and $p \in N$, while

$\Sigma_2 = \{2^n \cdot 3^m : n, m \in N \setminus \{0\}\}$ is a nonlacunal semigroup.

4.2. Definition. *Let $G \subseteq R$. Se shall say that G is discrete in the topology of R if for every $x \in G$ there exists an $\varepsilon > 0$ with $(x - \varepsilon, x + \varepsilon) \cap G = \{x\}$.*

4.3. Lemma. *Let $G \subseteq R$ be a nontrivial additive group such that the set G is discrete in the topology of R. Then there exists a real number u with the property*

$$G = uZ = \{uk : k \in Z\}.$$

Proof. Since G is discrete in the topology of R and G is an additive group, it follows that $u = \inf\{|x - y| : x, y \in G,\ x \neq y\} > 0$. From the properties if infimuum, it follows that there exist $x, y \in G$ such that $x - y = u$ and, consequently, $uZ \subseteq G$.

Let $x \in G$. We have

$$x = qu + r \text{ with } 0 \leqslant r < u,\ q \in Z.$$

If $r > 0$, then $u > r = x - qu \in G$ which is a contradiction. So, $r = 0$ and $x = qu \in uZ$ which completes the proof of the lemma.

The following two lemmas are immediate consequences of lemma 4.3.

4.4. LEMMA. *Let $G \subseteq R$ be a nontrivial additive subgroup of R. Then only one of the following assertions is true:*

(i) there exists a $u \in R \setminus \{0\}$ such that $G = uZ$

(ii) G is everywhere dense in R.

4.5. LEMMA. *Let $S \subseteq R_+$ be a nontrivial additive semigroup. If the additive group $G = S - S = \{s' - s'' : s', s'' \in S\}$ is discrete in the topology of R, then there exists a $u \in R \setminus \{0\}$ such that $S \subseteq G = uZ$.*

We notice that if S is a nontrivial countable additive semigroup, generated by the elements $a_1, a_2, \ldots \in S$, then

$$S - S = \bigcup_{n=1}^{\infty} (S - n(a_1 + \ldots + a_n)) \text{ and}$$

$$S - n(a_1 + \ldots a_n) \subseteq S - (n+1)(a_1 + \ldots + a_{n+1}),\ n \geqslant 1.$$

From this and from the foregoing lemmas, one can easily obtain the results of H. FURSTENBERG.

4.6. LEMMA (FURSTENBERG [3], lemma IV–1). *Let $\Sigma \subseteq Z$ be a multiplicative nonlacunal semigroup and suppose that*

$$\Sigma^+ = \{\sigma \in \Sigma : \sigma \geqslant 0\} = \{s_1, s_2, \ldots\} \text{ with } s_i < s_{i+1},\ i \geqslant 1.$$

Then

$$\lim_{n \to \infty} \frac{s_{n+1}}{s_n} = 1.$$

Lemma 4.6 gives a new class of sequences for which the results of section 2 are applicable.

The following theorem is a direct consequence of theorem 2.4. and lemma 4.6.

4.7. THEOREM. *Let $\Sigma \subseteq Z$ be a multiplicative nonlacunal semigroup and let $\Sigma^+ = \{\sigma \in \Sigma : \sigma \geqslant 0\} = \{s_1, s_2, \ldots\}$ with $s_i < s_{i+1}$ for $i \geqslant 1$. Then the sequence $(\ln s_n)_{n \geqslant 1}$ is a relatively dense universal for the class of all continuous periodical functions.*

We conclude with the following important result of H. FURSTENBERG [3].

4.8. LEMA. *Let $\Sigma \subseteq N$ be a multiplicative nonlacunal semigroup. Then, for every irrational number α, the set $\alpha\Sigma = \{\alpha\sigma : \sigma \in \Sigma\}$ is everywhere dense* (mod 1).

4.9. COROLLARY. Let $P = \{p_1, p_2, \ldots, p_n \ldots\}$ be the set of prime numbers and let $Q \subseteq P$ with card $(Q) \geqslant 2$. Then $\Sigma_Q = \{n \in N : p \text{ prime},\ p \mid n \Rightarrow p \in Q\}$ is evidently a multiplicative nonlacunal semigroup. Moreover, for every $k \in N \setminus \{0\}$, $\Sigma^k Q = \{n^k : n \in \Sigma_Q\}$ is also a multiplicative nonlacunal semigroup.

From the foregoing lemma, it follows that

$\alpha\Sigma_Q^k = \{\alpha n^k : n \in \Sigma_Q\}$ is everywhere dense (mod 1) for every irrational number α and $k \in N \setminus \{0\}$.

Remark that for every $\varepsilon > 0$, there exists a $Q \subseteq P$, Q infinite, with $\sum_{q \in Q} \frac{1}{q} < \varepsilon$.

The following two theorems are analogous to theorems 3.5 and 3.9. In their proofs, we use lemma 4.8 instead of lemma 3.4.

4.10. THEOREM. *Let $\Sigma = \{s_1, s_2, \ldots\}$ be a multiplicative nonlacunal semigroup and let $T \neq 0$ be a real number. For every real number α with the property that α/T is irrational, the sequence $(\alpha s_n)_{n \geqslant 1}$ is a relatively dense universal sequence for the class of continuous periodical functions with the period T.*

4.11. THEOREM. *Let $T \neq 0$ be a real number let $\alpha \in R$ with α/T irrational, let $\Sigma = \{s_1, s_2, \ldots\} \subseteq Z$ be a multiplicative nonlacunal semigroup and let $f : R \to R$ be a continuous function. If f has the property $f(\alpha s_n + mT) = f(\alpha s_n)$ for every $n \in N \setminus \{0\}$ and $m \in Z$, then f is a periodical function with the period T.*

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