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ON THE POPOVICIU CONVERSION OF JENSEN'S INEQUALITY

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1. Introduction. T. Popoviciu [1–3] proved some conversions of Jensen's inequality for the convex function $\varphi : [a, b] \to R$, where $0 \leqq a < b \leqq 1$, i.e. he proved the following results (see also [4, pp. 34–35]) :

THEOREM A. *Let $\varphi : [a, b] \to R$ be a convex function and let $f : [0,1] \to R$ be a continuous, nondecreasing and convex function such that $a \leqq f(x) \leqq b$ ($\forall x \in [0,1]$). Then*

$$(1)\quad \int_0^1 \varphi(f)\,\mathrm{d}x \leqq \frac{b+a-2A}{b-a}\,\varphi(a) + \frac{2(A-a)}{(b-a)^2}\int_a^b \varphi(x)\,\mathrm{d}x, \quad A = \int_0^1 f\,\mathrm{d}x.$$

If φ is strictly convex, the equality in (1) *is valid if and only if*

$$(2)\qquad f(x) = a + (b-a)\,\frac{x-\lambda+|x-\lambda|}{2(1-\lambda)}, \quad \lambda = \frac{b+a-2A}{b-a}.$$

THEOREM B. *Let $\varphi : [a, b] \to R$ be a continuous convex function and let $f : [0,1] \to R$ be convex of order $1, \ldots, n+1$ such that $a \leqq f(x) \leqq b$ ($\forall x \in [0,1]$). Then*

$$\int_0^1 \varphi(f)\,\mathrm{d}x \leqq \int_0^1 \varphi(U_j(A\,;x))\,\mathrm{d}x, \ \text{if}\ \frac{ja+b}{j+1} \leqq A \leqq \frac{(j-1)\,a+b}{j}, \quad 2 \leqq j \leqq n,$$

(3)

where

$$U_j(t\,;x) = a + j(j+1)\left(\left(t - \frac{ja+b}{j+1}\right) + \left(\frac{(j-1)\,a+b}{j} - t\right)x\right)x^{j-1},$$

(4)

and

$$(5)\qquad \int_0^1 \varphi(f)\,\mathrm{d}x \leqq V(A) \text{ if } a \leqq A \leqq \frac{na+b}{n+1},$$

where

$$(6)\qquad V(x)=\frac{b+na-(n+1)\,x}{b-a}\,\varphi(a)+\frac{(n+1)(x-a)}{n(b-a)\sqrt[n]{b-a}}\int_a^b\frac{\varphi(x)\,\mathrm{d}x}{\sqrt[n]{(x-a)^{n-1}}}.$$

If φ is strictly convex, the equality in (3) is valid if and only if

$$(7)\qquad f(x)=U_j(A;x),$$

and in (5) if

$$(8)\quad f(x)=a+(b-a)\left(\frac{x-\lambda+|x-\lambda|}{2(1-\lambda)}\right)^n,\quad \lambda=\frac{b+na-(n+1)A}{b-a}.$$

THEOREM C. *Let φ and f be defined as in Theorem A. Then*

$$(9)\quad A_\varphi-\varphi(A)\leqq\max_{x\in\left(a,\frac{a+b}{2}\right)}\left(\varphi(a)+\frac{2(x-a)}{(b-a)^2}\int_a^b(\varphi(t)-\varphi(a))\,\mathrm{d}t-\varphi(x)\right),$$

where $A_\varphi=\int_0^1\varphi(f)\,\mathrm{d}x.$

If φ is strictly convex, then the equality in (9) is valid for one value x_2 of x for the function

$$(10)\qquad f=a+(b-a)\,\frac{x-\lambda+|x-\lambda|}{2(1-\lambda)},\quad \lambda=\frac{b+a-2x_2}{b-a}.$$

In this paper, we shall show that Theorem C is a simple consequence of Theorem A, i.e. we shall give a generalization of Theorem C.

2. Generalization of the Popoviciu inequalities

THEOREM 1. *Let φ and f be defined as in* Theorem A, *and let $F: J^2\to R$ be nondecreasing in its first variable, where J is an interval such that $\varphi(x)\in J$ for all $x\in[a,b]$. Then*

$$F(A_\varphi,\ \varphi(A))\leqq\max_{x\in\left[a,\frac{a+b}{2}\right]}F\left(\frac{b+a-2x}{b-a}\varphi(a)+\frac{2(x-a)}{(b-a)^2}\int_a^b\varphi(x)\,\mathrm{d}x;\ \varphi(x)\right).$$

(11)

Proof. By (1) and the nondecreasing character of $F(.,y)$, we have

$$F(A_\varphi,\ \varphi(A))\leqq F\left(\frac{b+a-2A}{b-a}\varphi(a)+\frac{2(A-a)}{(b-a)^2}\int_a^b\varphi(x)\,\mathrm{d}x,\ \varphi(A)\right)$$

$$\leqq\max_{x\in\left[a,\frac{a+b}{2}\right]}F\left(\frac{b+a-2x}{b-a}\varphi(a)+\frac{2(x-a)}{(b-a)^2}\int_a^b\varphi(x)\,\mathrm{d}x,\ \varphi(x)\right)$$

since $A\in\left[a,\dfrac{a+b}{2}\right]$.

THEOREM 1′. Under the same hypotheses as in Theorem 1, except that F is nonincreasing in its first variable, we have

$$F(A_\varphi,\ \varphi(A))\geqq\min_{x\in\left[a,\frac{a+b}{2}\right]}F\left(\frac{b+a-2x}{b-a}\varphi(a)+\frac{2(x-a)}{(b-a)^2}\int_a^b\varphi(x)\,\mathrm{d}x,\ \varphi(x)\right).$$

(11′)

Remark. If $F(x,y)=x-y$, from (11) we get (9).

Similarly, using Theorem B we can get the following results:

THEOREM 2. *Let φ and f be defined as in Theorems B and F as in Theorem 1.*

(i) *If $(ja+b)/(j+1)\leqq A\leqq((j-1)a+b)/j\,(2\leqq j\leqq n)$ is valid, then*

$$(13)\qquad F(A_\varphi,\ \varphi(A))\leqq\max_{x\in\left[\frac{ja+b}{j+1},\ \frac{(j-1)a+b}{j}\right]}F\left(\int_0^1\varphi(U(x;\ t))\,\mathrm{d}t,\ \varphi(x)\right).$$

(ii) *If $a\leqq A\leqq(na+b)/(n+1)$, then*

$$(14)\qquad F(A_\varphi,\ \varphi(A))\leqq\max_{x\in\left[a,\ \frac{na+b}{n+1}\right]}F(V(x),\ \varphi(x)).$$

THEOREM 3. *Let φ and f be defined as in Theorem B and F as in Theorem 1. If*

$$(15)\qquad G(x)=\begin{cases}\displaystyle\int_0^1\varphi(U(x,t))\,\mathrm{d}t, & \text{for } (ja+b)/(j+1)\leqq x\leqq((j-1)a+b)/j,\ 2\leqq j\leqq n,\\ V(x), & \text{if } a\leqq x\leqq(na+b)/(n+1),\end{cases}$$

then

$$(16)\qquad F(A_\varphi,\ \varphi(A))\leqq\max_{x\in\left[a,\frac{a+b}{2}\right]}F(G(x),\ \varphi(x)).$$

Remark. We can also give two theorems similar to Theorem 1′.

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